CONSTRUCTING A METHOD FOR SOLVING THE RICCATI EQUATIONS TO DESCRIBE OBJECTS PARAMETERS IN AN ANALYTICAL FORM

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Виділено особливість нелінійних диференціальних рівнянь, які найбільш адекватно описують властивості об'єктів. Проаналізовано можливі методи їх лінеаризації. Відзначено проблеми, пов'язані з розв'язанням вихідних рівнянь у вигляді, які було лінеаризовано. Як приклад наведено рівняння Ріккаті.

Для рівняння Ріккаті спеціального виду розроблено метод його розв'язання з представленням результатів в аналітичному вигляді. В його основу закладено використання лінеаризації і спеціального методу обезрозмірювання.

Особливість розробленого методу визначається його використанням не до вихідного рівняння, а до його дискретного аналогу. Результатом розв'язання є аналітична залежність на основі елементарних функцій. Результат отримано на основі використання існуючого аналітичного розв'язання (опорного, базового) одного з рівнянь розглянутого типу. Всі вихідні рівняння розглянутого типу мають однотипне розв'язання. Це відноситься і до рівнянь, які не мали раніше аналітичного розв'язання.

Розроблено формалізовану процедуру реалізації розробленого методу. Вона дозволяє зв'язати аналітичний вид розв'язання даного рівняння з відомим аналітичним розв'язанням базового. Такий зв'язок можливий внаслідок рівності дискретних аналогів даного і базових рівнянь. Рівність дискретних аналогів забезпечується застосуванням спеціального методу обезрозмірювання.

Придатність методу і адекватність отриманих результатів показана шляхом їх порівняння з наявними аналітичними розв'язанням двох рівнянь Ріккаті спеціального виду. В одному випадку рішення має рухомі особливі точки. У другому випадку відоме рішення має асимптоту, але при додатних значеннях аргументу особливі точки відсутні.

Наведено можливість застосування розробленого методу для розв'язання рівняння Ріккаті загального вигляду

Ключові слова: рівняння Ріккаті, особливі точки, лінеаризація, обезразмеріваніе, аналітичне рішення, елементарні функції

1. Introduction

The features of most control objects can only be adequately displayed using the non-linear differential models. For this reason, finding a solution to them is an active area of research in modern management theory. Analytical precise methods can be applied to such equations in exceptional cases. The approximated methods are mostly used.

One of the provisions underlying the theory of automatic control (TAC) implies that whenever possible it is necessary to strive to build a model in a linear statement. This is because common solution methods are inherent only in the linear differential equations. Given this, the basic methods for solving non-linear differential equations are in their linearization.

There are various options for implementing this approach. Under TAC, in a general case, overcoming the non-linearity involves the expansion into Taylor's series by using the linear terms only. As a result, the transition from the original system to the linearised one entails some loss of accuracy. The model maintains adequacy relative to the original one only at small deviations from the point relative to which the expansion has been performed. Such a linearization technique can be considered not only as a special operation but rather as one of the ways to simplify the model.

The development of technology expands the scope of the application of automatic control instead of operator control. As a result, there is an increase in the range of deviations of managed magnitudes from the position of equilibrium.

In response to this challenge, new methods of linearization have emerged. Most of them are only applicable in particular cases for a certain type of non-linearities.

Although non-linear management theories and applications have made significant strides in recent years, there are still challenges in the areas of theoretical analysis and applied applications. In particular, this is manifested in solving the non-linear equations with moving special points. Therefore, it is a relevant task to develop new methods to analytically represent the results of solving nonlinear differential equations.

2. Literature review and problem statement

If one considers the non-linear models of different control objects as a single commonality, one should expect a unified approach to solving such models. This is the case when linearization is applied to solve them by expanding into Taylor's series and preserving the linear terms. However, the attempt to use other methods of linearization does not produce such a result. The proposed methods do not have a common methodological basis. They are applicable in individual particular cases or require serious mathematical transforms that make it difficult to apply them in practice. This thesis is confirmed by the publication of a special edition of the collection of papers [1] on the non-linear management systems. A large number of published articles underlie the need to find a link among research results in the considered areas.

A harmonic linearization method is used as one of the approaches to the approximate solution of non-linear systems [2]. It can be applied when it is possible to represent the examined system in the form of two parts: non-linear, to be linearized, and linear, which is the filter of all emerging harmonic components of the signal emerging at the output from a non-linear element, except the first. This design of the system under study has been observed in some cases in practice but is not comprehensive. In addition, the filtering capabilities of the linear part of the system depend on the parameters of the system and may not always provide satisfactory filtration quality.

Another linearization technique may be the application, to the original model of the transforms, inverse to the functions that are part of it. This is tantamount to the introduction of a new reference system, a new coordinate system. An example may be logarithmization. In relation to the original model, this technique can only be effective in rare cases of the functional homogeneity of all elements of the model [3]. More often, this technique applies to the results of solution [4], obtained, for example, by numerical or experimental means. In this case, following the linearization by solving an inverse problem, the original model can be obtained.

Another, more widely used, method is the linearization via feedback [5]. This method, similarly to logarithmization, applies the introduction of a new frame of reference, namely a new time dimension [6]. More to the point, its calculation, which makes it possible to represent the original nonlinear model in a linear form. This approach makes it possible to build regulators for non-linear systems similar to the linear ones. However, the resulting linearization algorithm is complex: its implementation requires a lot of information. Once the required amount of information is available, the values of the data acquired may vary slightly from each other. This leads to poor conditionality of the examined system with all the additional difficulties in solving. In addition, the required

amount of information is not always available. As a result, the number of objects suitable for control with the linearization via feedback is less compared to, for example, the Lyapunov function. The complexity of the task in question stimulates the search for other than direct methods of solving it. Thus, the possibility of using artificial neural networks to solve the problem of linearization via feedback is considered in [7]. The paper notes that 1,000 implementations were used to «train» the neural network. This is permissible when setting up robotics, which was considered in the cited paper. However, it may be unacceptable if 1,000 training (unmanageable) launches are needed, as is the case with, for instance, boiler equipment.

In the cases discussed above, the introduction of a new reference system was the tool that provided for the possibility of linearization. The use of modified unit systems can serve as a basis for simplifying the original model and thus simplifying its solution. The introduction of natural units as a method that makes it easier to simulate physical phenomena, similar to [8], has been used since at least 1881. In the cited article, based on physical constants (light speed, gravitational constant, and minimal electrical charge), the author introduced the first natural system of units (L – length, T – time, and M – mass). All other authors, both in previous years and now [9], have used an identical approach in building their unit systems. A certain set of physical constants is selected and an independent unit system is built on their basis. The idea is useful in the implementation of particular cases of the considered models, for which the appropriate set of physical constants is selected. For example, in theoretical physics, these kinds of unit systems sometimes simplify equations and gain some other benefits. However, due to the identity of the approach used, all cases demonstrate a uniform (the same) disadvantage. The natural unit systems used to date are inconvenient for practical measurements. The accuracy of unit reproduction is several orders of magnitude lower than the basic units of the International System (SI), as it is limited by the accuracy of knowledge of physical constants.

Paper [10] describes a method of introducing a natural unit system in the absence of the above-mentioned deficiencies. It is based on the operation of bringing the model to a nondimensional form. The appropriate procedure is performed to a deeper level than it is required by the standard method based on the Buckingham π theorem. In contrast to the generally accepted approach, the normalizing values are not selected from the list of characteristic values but are calculated considering the model in question and on its basis. The procedure is simple and does not go beyond algebraic transformations. The normalizing values are determined as the sets of parameters that correspond to certain processes described by the considered model. As the model changes, so do the type of normalizing values. The method, developed in [10], for introducing a system of natural values makes it possible to significantly reduce the number of model parameters without losing information. However, the cited paper does not consider the possibility of using the benefits obtained in the process of linearizing the model equations. This possibility is considered in work [11] using an example of studying the motion of a mathematical pendulum. The non-linearity of the model was determined by the possibility of taking into consideration the large initial deviation of the pendulum from the position of equilibrium (by 30°, 60°, and 90°) in the absence and presence of dissipative forces. The results were derived in the form of simple analytical expressions based on trigonometric and other elementary functions.

The range of their applicability was determined by achieving a deviation of the obtained results from the exact (numerical) values by 5 % (the accuracy of engineering calculations). The results of the calculations showed that for the case of harmonic oscillations, the deviations were absent across the entire range of the argument change. In the case of significant dissipative forces (no oscillatory process), the engineering accuracy of calculations is also ensured across the entire range of the argument change. For fading fluctuations, the engineering accuracy of calculations is maintained over three periods of oscillations (the authors understand the improper application of the term «period» for fading fluctuations). For comparison, in the case of linearization (without nondimensionalization, without the introduction of a natural system of units) only with the help of the first terms from Taylor's series, the engineering accuracy of the calculations is maintained even for the case of harmonic fluctuations:

– for the initial deviation of a pendulum by 30° – by 19 % of the duration of the first period from the point of linearization;

– for 60° – by 13 %;

– for 90° – by 9 %.

The results reported in [11] show the manifestation of a synergistic effect from combining a «standard» model linearization technology and using a special nondimensionalization method [10]. Moreover, one can talk about the emergence for such a combination of research tools. From these points of view, it is of interest to consider the possibility of applying them to differential equations, solving which is accompanied by certain difficulties, including those related to linearization. In particular, they include the Riccati equations that have breakpoints. It is important to be able to derive an analytical expression to represent the results of the solution. Especially so if this expression would take into consideration the character of the original equation and could be recorded by using elementary functions only.

3. The aim and objectives of the study

The aim of this study is to construct a method for solving a special type of Riccati equation to describe the parameters of objects using analytical expressions that reflect the physical essence of processes.

To accomplish the aim, the following tasks have been set:

– to develop a method for constructing an analytical expression to replace the numerical solution to the Riccati equation;

– to devise a methodology that reflects the physical essence of processes and formalizes the relationship between the analytical and numerical solutions to the Riccati equations;

– to find a solution to the Riccati equation that reflects the physical essence of processes.

4. A method for solving a special type Riccati equation with the analytical representation of results

4. 1. A method for building an analytical expression to replace the numerical solution to the Riccati equation

The methods for solving the Riccati equations were considered back in the first half of the 18th century when a series of mechanical tasks required the integration of an equation of the following form:

$$y' = p(x) \cdot y^2 + q(x) \cdot y + r(x). \tag{1}$$

The efforts of mathematicians, including Leibniz and the Bernoulli brothers, did not lead to success. Nothing has changed ever since. No general integration method has been developed, at least to express results using elementary functions. However, different approaches have been explored and solutions to a series of particular problems have been derived. An analysis of the current situation is given in [12].

To date, numerical solution methods, based on finite differences, have been considered as the most common. Along the way, there are issues both general, inherent in numerical solutions in general, and particular, inherent in solving the Riccati equations. Common issues relate to the complexity of generalizing and analyzing the results of discrete calculations compared to analytical ones. The issues of the second type are due to the presence of special points in the solution, in particular, the discontinuities of the second kind. The efforts described in this paper are aimed at resolving these issues.

The issues of the second type can be overcome by the methods developed so far. Widely used to construct the finite-difference schemes, the Euler method produces good results for smooth functions only. As one approaches a special point, the accuracy of the method decreases, the counting is increasingly unstable, and, eventually, the solution «falls apart». At the same time, the position of a special point cannot be determined with acceptable accuracy. Even more impossible is to continue the counting outside of it. By the year 2000, a practical method had been developed to determine the position of a particular point [13]. However, even in this case, it was impossible to continue counting beyond it. Finally, the task of passing in the numerical calculation of a special point and continuing to calculate beyond it without a noticeable loss of accuracy for the equations of form (1) is solved by the application of a calculation scheme obtained with a minor modification of the Euler analog and described in [12]:

$$y_{i+1} - y_i = \left(p_i \cdot y_i \cdot y_{i+1} + q_i \cdot y_i + r_i \right) \cdot \Delta x_i. \tag{2}$$

Here, Δx is the step in changing the argument; $i+1$ and i are the indices that denote the calculated values and taken from the preceding calculation step, respectively.

One of the features of analog (2) is its linearity relative to the values it includes, despite the quadratic dependence in the right-hand side of original equation (1). In fact, (2) is the current linearization of original equation (1) at the beginning x_i of each sampling step Δx_i. The use of this property in conjunction with a special nondimensionalization method [10] underlies the construction of an analytical expression to represent numerical calculation results.

Consider solving a special type ($q(x)=0$) Riccati equation in the following form:

$$y' = p(x) \cdot y^2 + r(x). \tag{3}$$

According to (2), its discrete analog takes the following form:

$$y_{i+1} = \frac{y_i + r_i \cdot \Delta x_i}{1 - p_i \cdot y_i \cdot \Delta x_i}. \tag{4}$$

The solution is based on using the available precise solution to another special type Riccati equation:

$$y' = y^2 + 1 \Rightarrow y = \mathrm{tg}(x), \tag{5}$$

whose discrete analog is written as follows:

$$\bar{y}_{i+1} = \frac{\bar{y}_i + \Delta\bar{x}_i}{1 - \bar{y}_i \cdot \Delta\bar{x}_i}. \tag{6}$$

When resolving technical tasks, the nondimensionalization procedure is performed for the original equations of the mathematical model. To solve the set problem, we apply it to the discrete analog (4). At every step of the calculation, the values p_i, r_i – const and can be considered as process parameters at this step. Denote:

$$y = \bar{y} \cdot y^\Delta, \quad \Delta x = \Delta\bar{x} \cdot x^\Delta. \tag{7}$$

Here $\bar{y}$, $\Delta\bar{x}$ are the normalized values, y^Δ, x^Δ are the normalizing values. Substitute (7) into (4):

$$\bar{y}_{i+1} = \frac{\bar{y}_i + \left(\dfrac{r_i \cdot x_i^\Delta}{y_i^\Delta}\right) \cdot \Delta\bar{x}_i}{1 - \left(p_i \cdot y_i^\Delta \cdot x_i^\Delta\right) \cdot \bar{y}_i \cdot \Delta\bar{x}_i}. \tag{8}$$

By applying the procedure in accordance with the method reported in [10] to (8), we obtain the normalizing values y^Δ and x^Δ, which transform expressions in brackets to «1» and, accordingly, convert it into identical to (6):

$$x_i^\Delta = \sqrt{\frac{1}{p_i \cdot r_i}}, \quad y_i^\Delta = \sqrt{\frac{r_i}{p_i}}. \tag{9}$$

Thus, the discrete analog (6) for equation (5) is a nondimensionalized form of discrete analog (4) for equation (3). In other words, the solution to equation (3) can be obtained from the solution to (5), by converting it using the normalizing values (9). At the same time, equation (5) has a simple analytical solution: $y = \mathrm{tg}(x)$.

The idea of representing a solution to equation (3) in the analytical form is to formalize its relationship with the analytical solution (5) through their discrete analogs, coinciding in the nondimensionalized form.

To demonstrate the procedure for formalizing this relationship, consider a special type Riccati equation in the following form:

$$y' = y^2 + x^2. \tag{10}$$

Such a choice is predetermined by the presence of a precise analytical solution based on Bessel's functions:

$$y = -\frac{x \cdot \left[J_{-\frac{3}{4}}\left(\dfrac{1}{2}x^2\right) - Y_{-\frac{3}{4}}\left(\dfrac{1}{2}x^2\right)\right]}{J_{\frac{1}{4}}\left(\dfrac{1}{2}x^2\right) - Y_{\frac{1}{4}}\left(\dfrac{1}{2}x^2\right)}. \tag{11}$$

This will make it possible to assess the accuracy of the result to be obtained. Solutions such (11) are only possible for a limited type of equations (3):

$$y' = b \cdot y^2 + c \cdot x^\alpha, \quad b, c, \alpha = \text{const},$$

$$\alpha = -\frac{4 \cdot k}{2 \cdot k \pm 1}, \text{ for } k = 0, \pm 1, \pm 2, \ldots$$

The proposed solving method does not impose restrictions on the form $p(x)$ and $r(x)$.

4. 2. A procedure that reflects the physical essence of processes, formalizes the relationship between analytical and numerical solutions to Riccati equations

For equation (10), in its general form (3), we obtain $p(x) = 1$ and $r(x) = x^2$. At the same time, based on (4), the discrete analog takes the following form:

$$y_{i+1} = \frac{y_i + x_i^2 \cdot \Delta x_i}{1 - y_i \cdot \Delta x_i}. \tag{12}$$

Denote Δx_i, x_i, y_i, y_{i+1} are the values from a discrete analog (12), $\Delta\bar{x}_i$, $\bar{x}_i$, $\bar{y}_i$, $\bar{y}_{i+1}$ are the values of the discrete analog (6), which serves as a nondimensionalized version of analog (12). They are related according to (7), where, from (9), the following values are the normalizing ones:

$$x_i^\Delta = \frac{1}{x_i}, \quad y_i^\Delta = x_i. \tag{13}$$

The procedure for relating the relevant values of discrete analogs (12) and (6) is as follows:

1) the range of change in the argument of the examined function is split into intervals (steps). In a general case, the intervals can be of varying sizes. In the examined case (for equation (10)), it is accepted:
– a range of change in the argument $x_i \in 0\ldots3.6$;
– a step of calculating $\Delta x_i = 0.025$ – const;

2) according to (7) and considering (13), the step of calculation for (6) is determined from the following ratio:

$$\Delta\bar{x}_i = \frac{\Delta x_i}{x_i^\Delta} = \Delta x_i \cdot x_i, \tag{14}$$

and the argument's current value:

$$\bar{x}_i = \bar{x}_{i-1} + \Delta\bar{x}_i; \tag{15}$$

3) for the current value $\bar{x}_i$, by using (6), the corresponding value $\bar{y}_{i+1}$ is defined. The specificity of the calculation of $\bar{y}_i$ for this expression needs to be taken into consideration. Due to the recurrent nature of (6), as $\bar{y}_i$, the current step of the calculation employs $\hat{\bar{y}}_{i+1}$ from the preceding one. In doing so, each step $\Delta\bar{x}_{i-1}$, $\Delta\bar{x}_i$ of changing the argument according to (7) is calculated from Δx_{i-1}, Δx_i using a variable normalization by values x_{i-1}, x_i. For this reason, in the transition from $\hat{\bar{y}}_{i+1}$ to $\bar{y}_i$ their equality is not satisfied. Based on the nature of normalizations (7), (13), the transition follows the following expression:

$$\bar{y}_i = \hat{\bar{y}}_{i+1} \cdot \frac{x_{i-1}}{x_i}; \tag{16}$$

4) according to (7) and (13), the value y_i is determined from the following ratio:

$$y_{i+1} = \bar{y}_{i+1} \cdot y_i^\Delta = \bar{y}_{i+1} \cdot x_i. \tag{17}$$

The validity of the described transformations is confirmed by the results of calculations shown by the charts given in Fig. 1.

These are the results of solving equation (10) in three ways. The first was obtained on the basis of analytical solution (11). A numerical solution based on a discrete analog (12) was used to derive a second solution. The third chart

shows the results obtained using a numerical solution based on a discrete analog (6) for equation (5) with a further recalculation of the results in line with the described algorithm. The convergence of charts indicates that the results of the calculations using all methods coincide.
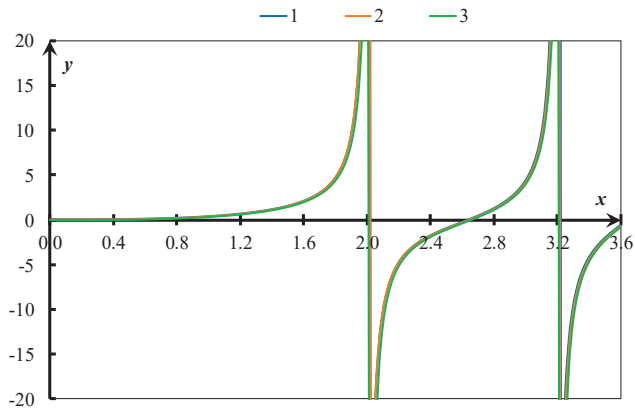

Fig. 1. Results of solving equation (10):
1 — analytical, based on (11); 2 — numerical, using a discrete analog (12); 3 — numerical, using a discrete analog (6) with further recalculation of results

4. 3. A solution to the Riccati equation that reflects the physical essence of processes

Comparing the analytical solution $y=\mathrm{tg}(x)$ to equation (5) with the results of its numerical solution using a discrete analog (6) produces a good qualitative and quantitative match when passing through several special points, similar to the one shown in Fig. 1. After using (6) in the procedure described in 4.3, comparing similar values does not give such a match. This is due to the use in the discrete analog of the variables normalized (16) in a special way. However, the nature of change in the values from (6) and $y=\mathrm{tg}(x)$ remains similar.

Assuming that, after nondimensionalization, the values derived from (6) are the result of solving, at each point, the equation $\bar{y}_i = \mathrm{tg}(\bar{x}_i)$, we perform the inverse transforms:

$$\overset{\dots}{\bar{x}}_i = \mathrm{arctg}(\bar{y}_i) \tag{18}$$

and compare the values obtained with the corresponding x_i (Fig. 2).

$$y = 0.0101x^3 + 0.5088x^2 - 0.2915x + 0.0693$$
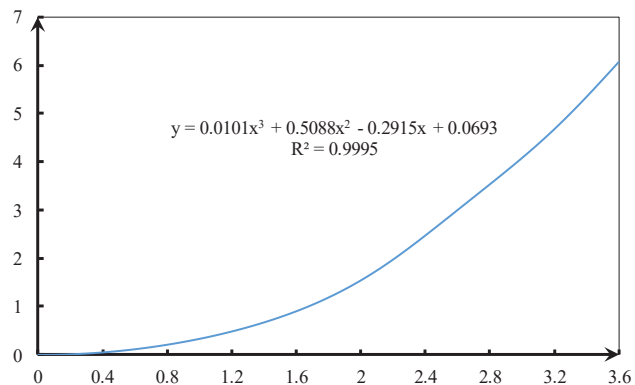$$R^2 = 0.9995$$

Fig. 2. The ratio of the arguments x_i of the initial function to the result of transforming (18) a nondimensionalized expression (6)

The resulting dependence is represented by a smooth curve and is well-approximated by the polynomials of various powers. Their types are shown in Fig. 2. The reliability of the approximation by a second-power polynomial $\left(_2 R^2 = 0.9995\right)$ is almost the same as the similar characteristic of a fifth-power polynomial $\left(_5 R^2 = 0.9998\right)$. This determines the possibility of using a polynomial of the minimum (second) power to approximate the results (Fig. 2).

To sum up the above, the analytical dependence to represent the results of solving equation (10) can be obtained as a result of the following sequence of actions:

1) a polynomial (Fig. 2) is used to determine the value $\overset{\dots}{\bar{x}}_i$, that corresponds to the predefined value x_i;

2) the expression $\bar{y}_i = \mathrm{tg}(\overset{\dots}{\bar{x}}_i)$, which is inverse to (18), is used to determine $\bar{y}_i$ from the normalized solution (6);

3) taking into consideration the normalizing expression (17) is used to determine a solution to the desired equation (10).

The resulting expression takes the form:

$$y = x \cdot \mathrm{tg}(0.5634x^2 - 0.3699x + 0.0924). \tag{19}$$

The validity of the representation of the results of solving equation (10) applying expression (19) is shown by comparing them with the results of precise solution using (11) (Fig. 3).
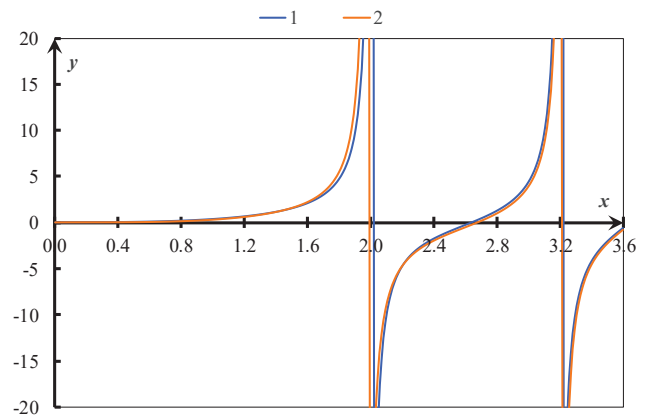

Fig. 3. The results of solving equation (10):
1 — precise, based on expression (11); 2 — approximate, based on expression (19)

The comparison of the solving results shows their good quality and quantitative agreement. At the same time, the examined interval of the argument change includes two special moving points.

5. Discussion of results of constructing a method to solve the special type Riccati equations

Papers [10, 11] noted a synergistic effect when solving by combining the procedures of linearization and a special method for the nondimensionalization of a mathematical model. In practice, the linearization procedure applies to the original equations and approximates them properly in a narrow range of argument changes. As a result, a solution obtained on this basis adequately reflects the behavior of a mathematical model at a small change in the arguments relative to the point of linearization. The use of a special nondimensionalization method [10] significantly extends this range, but the limitations remain. The result gets worse in proportion to the nonlinearity of the original model. Hence, a peculiar paradox: under a larger non-linearity of the original model, the behavior of its solution is relevant over a wider range of

arguments changes. However, it is in this case that the applied linearization methods yield a smaller range of variations in arguments for an adequate solution. The problem almost defies a solution when it is necessary to overcome special points (discontinuities) in solutions.

The method proposed in this work is also based on the combined application of linearization and a special nondimensionalization method, but for a different form of a model, to its discrete analog. The analog is built as a set of models based on the linearization of the original equations. In this case, one can argue about the applicability of the accepted linearization parameters to the entire range of changes in the model's arguments determined at the calculation step under consideration.

Another feature of the proposed method is the use of dualism in the way of solving some of the equations from the list under consideration. In this case, it is an equation (5) with the possibility of its analytical solution in the form $y=\mathrm{tg}(x)$ and numerical one in form (6). The application of a special nondimensionalization procedure makes it possible to bring a discrete analog (4) of any equation from the list under consideration (in this case, (3) to the form (6) for equation (5).

The third feature of the method used is the employment of a discrete analog for the case in question, making it possible to conduct through calculation before and after special points (points of discontinuities) with a margin of error that does not violate the adequacy of calculations.

Emergence in the use of the specified features is manifested in the possibility of analytical representation of the results of solving the equations in the form (3). This applies to all types of such equations, including those that do not have an analytical solution in the classical form.

This work distinguishes the concepts of analytical solution and the representation of the results of numerical solutions in an analytical form. In the present case, the type of approximation dependence is not selected from a series of those available but is built in accordance with the nature of the process in question. That makes it possible to take into consideration the specificity of the processes under study and, as a result, to correctly represent them. Thus, the built expression (19) for a special type (10) Riccati equation makes it possible to determine the position of special points in the end-to-end calculation. This correctly represents the results of the calculation in the entire range of change in the argument, including in the regions of special points.

When assessing the validity of approximation, one is limited to a quantitative comparison of the original data with the results of the calculation using approximation curves. The qualitative (physical) aspect of the processes described is generally not considered. The type of the original equation (10), chosen in our work, demonstrates the broader possibilities of the proposed method compared to a standard approximation scheme. The structure of the derived expression (19) corresponds to the structure of the expression of a precise solution (11). Both expressions consist of the product of an argument by a function. The coincidence of the structures of the equations is one of the reasons for the good qualitative and quantitative match of the calculation results (Fig. 3). The equations differ in the form of recording an efficient as a function. For the case of an analytical solution (11), it is a set of the Bessel functions of the first and second kinds of fractional orders. In expression (19), the efficient is built using the elementary functions of tangent and a second-power polynomial as its argument.

The generally accepted approach to analytical solutions to equations leads to the consideration of both basic and insignificant effects. In practice, the raw data are always determined with some error. For this reason, accounting for small effects is not practical but, in many cases, does not make it possible to derive solutions or affect its form and complexity. This is also the case for the same-type original equations. As noted above, it is not always possible to derive an analytical solution to the equations of form (3). And if it exists, it acquires a different and often complex form. Thus, for (5), the solution is $y=\mathrm{tg}(x)$, for (10) – expression (11), recorded with the help of special functions (Bessel functions), and, for example, for the equation:

$$y' = \frac{1}{2}y^2 + \frac{1}{2x^2}, \tag{20}$$

the equation in the form:

$$y = -\frac{1}{x} \cdot \left(1 - \frac{2}{C - \ln|x|}\right). \tag{21}$$

The proposed method is based on a different approach. When applying it to a certain type of equation (in the considered case, (3)), a supporting (basic) solution to the same type of equation (in the considered case, (5)) is used. Its feature is the presence of a simple analytical solution, as well as the representation of the solution on the basis of a discrete analog (6). Next, the analytical representation of the solution to all equations of the type in question is recorded based on a basic solution. In this case, the form of equations solved is used only to determine the normalizing values (9). Under this approach, the general form of the solution to the special type Riccati equations is represented as:

$$y = y^{\Delta} \cdot \mathrm{tg}\left[f\left(\bar{x}\right)\right], \ \ \bar{x} = x/x^{\Delta}. \tag{22}$$

Thus, it follows from (5), $x^{\Delta}=1$, $y^{\Delta}=1$, and, taking into consideration (22), the solution (basic) takes the following form $y=\mathrm{tg}(x)$. From (10) $\rightarrow x^{\Delta}=1/x$, $y^{\Delta}=x$ and a solution of form (19). From (20) $\rightarrow x^{\Delta}=2x$, $y^{\Delta}=1/x$ and a solution in the following form:

$$y = \frac{1}{x} \cdot \mathrm{tg}\binom{-0.319x^6 + 2.181x^5 - 5.883x^4 +}{+7.951x^3 - 5.663x^2 + 2.085x - 1.355}. \tag{23}$$

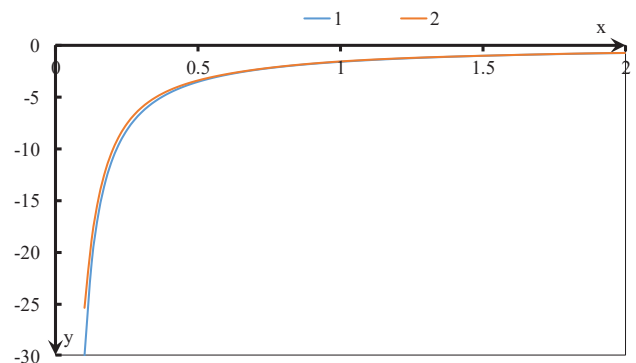The comparison of calculation results using equations (21) and (23) are shown in Fig. 4.

Fig. 4. The results of solving equation (20):
1 — precise, based on expression (21); 2 — approximate, based on expression (23)

In (21), it is adopted $C=-3.3$. The initial conditions for calculating this value are determined from the sampling step when deriving a support solution.

In the latter case, expression (23) for an approximate solution takes a more complex form than an accurate solution (21). However, even in this case, the possibility of obtaining the same-type adequate solution to the special type Riccati equations using a single method has been demonstrated. This possibility applies to all equations of a given type, including those without an analytical solution.

The possibility of using the proposed method can be considered for the general Riccati equations. At $p(x)=1$, $q(x)=1$, $r(x)=1$, equation (1) is reduced to an equation with separating variables:

$$y' = y^2 + y + 1. \tag{24}$$

It has an analytical solution that can be used as a supporting (basic) one.

6. Conclusions

1. A method for representing a solution to the special type Riccati equation in analytical form has been developed. The method is based on representing, using a special nondimensionalization method, a relationship among the discrete analogs of various Riccati equations. One of these equations is supporting and has a simple analytical solution using an elementary function. The method does not impose restrictions on the form of the coefficients of the equation under study.

2. A procedure has been devised to formalize the construction of the relationship of the analytical solution to the supporting equation through its discrete analog of a special form with a discrete analog of the studied equation to be solved. This makes it possible, by using an analytical solution to the supporting equation, to build an analytical expression to represent the behavior of the solution to the desired equation.

3. An analytical expression has been obtained to represent the behavior of a solution to the examined special type Riccati equation. Its feature is the simplicity, determined by using a minimum number of elementary functions in the expression. A distinctive feature is the ability to determine the position of special points (points of solution discontinuities) and to perform calculations in the surrounding area. The adequacy of the resulting solution has been shown by comparing its results with the results of analytical solutions to various special type Riccati equations. Using an example of the special type Riccati equation, the possibility of describing their solutions using the same-type equation based on elementary functions has been demonstrated. A possibility has been shown to find a solution to the general Riccati equations by using the developed method.

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