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RECIPROCAL RELATIONS FOR THE OPEN HYDRODYNAMIC STEADY STATES (OHSS)

Based on Onsager's regressive hypothesis and a local equilibrium in hydrodynamics, the time symmetry of the mutual correlation functions of fluctuations is analyzed directly from the macroscopic equations of motion for the open steady state of a continuous medium. It is shown that, in OHSS, the flux violates the symmetry of correlation functions and Onsager's reciprocal relation, which take place near the equilibrium steady state. The reciprocal relations for OHSS are found. Examples of their use are considered.

Keywords: fluctuations, open hydrodynamic steady states, reciprocal relations.

1. Introduction

In work [1], a general approach to solve the problem of hydrodynamic fluctuations in steady states due to external fields such as a temperature gradient and so on, was considered. The presence of flows caused by the fields makes the corresponding states open, and violates the time symmetry of fluctuations, which was first noticed by Vladimirsky [2]. Namely this violation is the main feature of fluctuations in systems with flow. The ideological basis of the developed theory, together with Onsager's regression hypothesis about the dynamics of fluctuations, was the local equilibrium. The local equilibrium in hydrodynamics, where the medium is considered as continuous, well known for a long time. The microscopic substantiation for this was done by Bogolubov [3, 4], who showed that the evolution of a system of molecules is divided into the kinetic and hydrodynamic stages. The hydrodynamic stage is achieved over a time interval required for several collisions of molecules. After that, a local equilibrium is established, and one can speak only about macroscopic hydrodynamic variables, which depend on the spatial-temporal coordinates, and their fluctuations. For gases, the boundary of the kinetic and hydrodynamic stages is 10^{-9} s, for liquids – even less (10^{-12}–10^{-13} s). Thus, accepting Onsager's regression hypothesis describing the fluctuations by macroscopic (hydrodynamic) equations, we must assume that the stochastic properties of fluctuations should be determined by the locally equilibrium distribution function. Onsager's hypothesis and the local equilibrium distribution function for an open state define the Ornshtein–Uhlenbeck process and uniquely determine the fluctuating forces in the problem the second or Langevin fluctuation-dissipation theorem (FDT). So, the calculation of the correlation functions of fluctuations can be made either by solving the homogeneous Cauchy problem for a linear system of hydrodynamic equations with the subsequent averaging of initial conditions or by solving a non-uniform system of the same equations with added fluctuating forces with the subsequent averaging over forces. The result will be the same. I would like to draw attention that fluctuating forces are fictitious. It is just a mathematical method of solving the same problem with the initial conditions [5]. In the works of other authors on this subject, the emphasis was placed on the fact that the states with a flow are non-equilibrium, and the main problem is to find the distribution function for hydrodynamic fluctuations (see [6] and references therein). The term NESS (non-equilibrium steady state) is introduced and, by kinetic methods, makes sure that only Lagevin description of fluctuations in such systems with the forces obtained by Landau and Lifshitz for equilibrium fluids [7] should be used. For the first time, these forces were applied by Zaitsev and Shliomis [8] for the Rayleigh–Bénard problem. Later, such approach was named fluctuating hydrodynamics. It was actively developed by the authors of book [9] and the recent publication [10]. According to the fluctuating hydrodynamics, the fluctuating forces are determined not by the dynamic matrix of the considered state, but by a matrix corresponding to the dynamics of fluctuations near the equilibrium state of a fluid, where there are no flows. That is the forces that are not relevant to the problem are used. The Langevin FDT used in [1] does not require any kinetic substantiation. In order to emphasize the lack of the need to apply kinetic methods to find the distribution function for hydrodynamic fluctuations, we use the term "open hydrodynamic steady state" (OHSS) instead of the term "non-equilibrium steady state" (NESS). This means that we consider the hydrodynamics, where the non-equilibrium is reduced to a local equilibrium. A violation of the time symmetry by a flow leads, obviously, to a violation of Onsager's reciprocal relations. The goal of this work is to find appropriate relations for the OHSS.

2. Reciprocal Relations in the Absence and Presence of a Flow

In 1931, Onsager [11, 12] established the reciprocal relations for kinetic coefficients

$$\gamma_{ij} = \gamma_{ji}, \tag{1}$$

which play an important role in thermodynamics. Here and below, we will use the notation adopted in [5]. The kinetic coefficients are equal

$$\gamma_{ij} = \lambda_{ik}\beta^{-1}_{kj}, \tag{2}$$

where the matrix λ determines the dynamics of fluctuations around the initial steady state,

$$\dot{x}_i = -\lambda_{ik}x_k, \tag{3}$$

and the matrix β sets the distribution function for initial fluctuations

$$f(\mathbf{x}) \propto \exp\left(-\frac{1}{2}\beta_{ij}x_i x_j\right). \tag{4}$$

Onsager showed that (1) follows from the principle of detailed balance, which is expressed by the symmetry of the mutual correlation functions of fluctuations with respect to the time:

$$\langle x_i\,(t)\,x_j\rangle = \langle x_j\,(t)\,x_i\rangle. \tag{5}$$

This symmetry is explained by the fact that the equations of motion of molecules of a substance are symmetric with respect to the replacement of the sign of the time. This explanation is nowhere mathematically used and exists only as a clarification. In fact, relation (5) is the starting point: if we differentiate (5) with respect to the time, then use (3), and put $t = 0$, we will come to (1). Formulas (1) and (5) are valid, when both variables x have the same parity. When the parity of variables is different, the correlation functions are odd, and the sign $\pm$ on the right should be placed in formulas.

Since we are talking about hydrodynamic fluctuations, the reference to the microscopic equations of motion of molecules is unsatisfactory, because the kinetic and hydrodynamic stages, according to [3], are separated. Due to this circumstance, we should give another explanation to the detailed balance in hydrodynamics that would not require a reference to the kinetics.

We can use the stationarity of the process and write (5) in the form

$$\langle x_i\,(t)\,x_j\rangle = \pm\,\langle x_i\,(-t)\,x_j\rangle. \tag{6}$$

It means that correlation functions of fluctuations are even or odd, when the detailed balance is satisfied. This can only be in a flowless state, that is, in the equilibrium. Thus, the macroscopic (hydrodynamic) explanation of relation (5) is the absence of flows in the system. In other words, the initial (zero) state in the Onsager theory is equilibrium. For the OHSS, the principle of detailed balance (5) does not take place, which means that the reciprocal relation (1) will be broken.

To obtain the relations of reciprocity for such systems, we turn directly to the macroscopic equations of motion for fluctuations. Let us consider the case of two cross-processes, so that the fluctuation deviations from OHSS are described by the relaxation equations

$$\begin{aligned}\dot{x}_1 &= -\lambda_{11}x_1 - \lambda_{12}x_2,\\ \dot{x}_2 &= -\lambda_{21}x_1 - \lambda_{22}x_2.\end{aligned} \tag{7}$$

The coefficients λ_{12} and λ_{21} are assumed to be nonzero.

Let us solve the Cauchy problem for (7):

$$\begin{aligned}&x_1(t) = \{x_1[(\lambda_2+\lambda_{11})e^{\lambda_1 t} - (\lambda_1+\lambda_{11})e^{\lambda_2 t}] + \\ &+ \lambda_{12}x_2(e^{\lambda_1 t} - e^{\lambda_2 t})\}/(\lambda_2-\lambda_1),\\ &x_2(t) = \{x_2[-(\lambda_1+\lambda_{11})e^{\lambda_1 t} + (\lambda_2+\lambda_{11})e^{\lambda_2 t}] + \\ &+ \lambda_{21}x_1(e^{\lambda_1 t} - e^{\lambda_2 t})\}/(\lambda_2-\lambda_1),\end{aligned} \tag{8}$$

where λ_1 and λ_2 are eigenvalues of system (7).

Using of the condition of orthogonality of the initial values of coordinates x and forces $X(X_i = \beta_{ik}x_k$ [5]),

$$\langle x_i X_j\rangle = \delta_{ij}, \tag{9}$$

we'll find the following reciprocal relation from (8)

$$\frac{\langle x_1(t)\,X_2\rangle}{\langle x_2(t)\,X_1\rangle} = \frac{\lambda_{12}}{\lambda_{21}} \tag{10}$$

or

$$\langle J^x_{12}(t)\,X_1\rangle = \langle J^x_{21}(t)\,X_2\rangle, \tag{11}$$

where $J^x_{12} = -\lambda_{12}x_2$ and $J^x_{21} = -\lambda_{21}x_1$ are the nondiagonal fluctuation flows in (7). The index x is introduced for the distinction of flows defined as the products of kinetic coefficients γ and forces X.

Carrying out the Fourier transformation in (11), we have

$$(J^x_{12}X_1)_\omega = (J^x_{21}X_2)_\omega. \tag{12}$$

Equation (12) expresses the following theorem: the average power at a frequency ω of the force X_1 on the flow J^x_{12} is equal to the average power of the force X_2 on the flow J^x_{21} at the same frequency. The theorem is similar to the theorem on the reciprocity of works in mechanics (the Maxwell–Betti reciprocal theorem), according to which the work carried out by the force F_1 on the displacement δ_{12} caused by the force F_2 is equal to the work carried out by the force F_2 on the displacement δ_{21} caused by the action of the force F_1.

If the forces are statistically independent ($\beta_{12} = 0$, $\gamma_{12} = \lambda_{12}\beta_{22}^{-1}$, $\gamma_{21} = \lambda_{21}\beta_{11}^{-1}$), then (10) is transformed into

$$\frac{\langle x_1(t)\,x_2\rangle}{\langle x_2(t)\,x_1\rangle} = \frac{\gamma_{12}}{\gamma_{21}}. \tag{13}$$

Thus, the mutual correlation functions of fluctuations are related as the corresponding kinetic coefficients. Formula (13) can be written so that it expresses a time symmetry violation of the mutual correlation function in the OHSS:

$$\frac{\langle x_1(t)\,x_2\rangle}{\langle x_1(-t)\,x_2\rangle} = \frac{\gamma_{12}}{\gamma_{21}}. \tag{14}$$

It is clear that the choice of statistically independent coordinates and forces can always be carried out, since the matrix β is symmetric. It is easy to write down the reciprocal relation in the general case $\beta_{12} \neq 0$. However, it becomes more cumbersome and requires a separate discussion.

3. Unbounded Liquid with Temperature Gradient

Let us illustrate the above-stated by examples of the OHSS previously discussed by the author.

An unbounded liquid with a stationary temperature gradient $T = T_0 + \mathbf{r}\boldsymbol{\nabla} T$ is open due to a stationary heat flux through it. Fluctuation disturbances in hydrodynamics should be expanded into a series of modes that satisfy the boundary conditions. In the case of unbounded liquid, we have the following Fourier series for density fluctuations ρ and for the potential of velocity fluctuations φ of the longitudinal sound:

$$\lambda = \begin{pmatrix}\rho\\ \varphi\end{pmatrix} = \frac{1}{\sqrt{V}}\sum_{\mathbf{k}}\begin{pmatrix}\rho_{\mathbf{k}}\\ \varphi_{\mathbf{k}}\end{pmatrix}\exp(i\mathbf{kr}), \tag{15}$$

where V is the volume, for which we make normalization. The amplitudes of the modes or Fourier components play the role of variables x_i. Their dynamics is determined by the matrix (see [1])

$$\lambda = \begin{pmatrix}0 & -\rho_0 k^2\\ c^2/\rho_0 & Dk^2\end{pmatrix}, \tag{16}$$

where c is the speed of sound, ρ_0 is the liquid density, $D = \frac{4}{3}\nu + \xi$, ν and ξ are the kinematic viscosities, and the simultaneous local equilibrium correlation functions are statistically independent and equal to

$$\beta^{-1} = \begin{pmatrix} \langle \rho_{\mathbf{k}} \rho_{\mathbf{k}'} \rangle & 0 \\ 0 & \langle \varphi_{\mathbf{k}} \varphi_{\mathbf{k}'} \rangle \end{pmatrix} = \\ = \begin{pmatrix} \dfrac{\rho_0}{c^2} & 0 \\ 0 & -\dfrac{\mathbf{k}\mathbf{k}'}{\rho_0 k^2 k'^2} \end{pmatrix} T_0 \Delta_{\mathbf{k},\mathbf{k}'}, \tag{17}$$

where

$$\Delta_{\mathbf{k},\mathbf{k}'} = \delta_{\mathbf{k},-\mathbf{k}'} + \frac{i}{2}\left(\delta_{\mathbf{k}+\mathbf{q},-\mathbf{k}'} - \delta_{\mathbf{k}-\mathbf{q},-\mathbf{k}'}\right), \ \mathbf{q} = \frac{\boldsymbol{\nabla} T}{T_0}. \tag{18}$$

For the kinetic coefficients, we have

$$\gamma = \lambda \beta^{-1} = \begin{pmatrix} 0 & \dfrac{\mathbf{k}\mathbf{k}'}{k'^2} \\ 1 & -\dfrac{D\mathbf{k}\mathbf{k}'}{\rho_0 k'^2} \end{pmatrix} T_0 \Delta_{\mathbf{k},\mathbf{k}'}. \tag{19}$$

As can be seen, Onsager's reciprocal relation $\gamma_{1,\mathbf{k};2,\mathbf{k}'} = -\gamma_{2,\mathbf{k}';1,\mathbf{k}}$ is not satisfied, which is a result of the violation of the principle of detailed balance by the heat flux. Indeed, from the dynamic equations with matrix (16), we get

$$\langle \rho_{\mathbf{k}}(t)\, \varphi_{\mathbf{k}'} \rangle = -\frac{T_0}{ck} \frac{\mathbf{k}\mathbf{k}'}{k'^2} \Delta_{\mathbf{k},\mathbf{k}'} e^{-\frac{1}{2}Dk^2 t} \sin(ckt), \tag{20}$$

$$\langle \varphi_{\mathbf{k}}(t)\, \rho_{\mathbf{k}'} \rangle = -\frac{T_0}{ck} \Delta_{\mathbf{k},\mathbf{k}'} e^{-\frac{1}{2}Dk^2 t} \sin(ckt). \tag{21}$$

It can be see that $\langle \rho_{\mathbf{k}}(t)\, \varphi_{\mathbf{k}'} \rangle \neq -\langle \varphi_{\mathbf{k}'}(t)\, \rho_{\mathbf{k}} \rangle$ At the same time, taking the ratio (20) to (21), we obtain (13).

4. Convective Rayleigh–Bénard Instability

In the case of the convective Rayleigh–Bénard instability for the layer of an incompressible fluid, the system includes the macroscopic fields of temperature and gravity. In the Boussinesq approximation for a layer with free boundaries, the equations for the spatial Fourier components of the vertical velocity $w_{\mathbf{k}}$ and temperature fluctuations $\theta_{\mathbf{k}}$ are as follows [13]:

$$\begin{cases} \dfrac{\partial w_{\mathbf{k}}}{\partial t} = -\nu k^2 w_{\mathbf{k}} + \alpha g \dfrac{\kappa^2}{k^2} \theta_{\mathbf{k}}, \\ \dfrac{\partial \theta_{\mathbf{k}}}{\partial t} = \beta w_{\mathbf{k}} - \chi k^2 \theta_{\mathbf{k}}, \end{cases} \tag{22}$$

where $\mathbf{k}(\kappa, q) = \mathbf{k}\left(\kappa, \frac{n\pi}{l}\right)$, n is an integer, l is the layer thickness, vertical temperature gradient equals $-\beta \mathbf{z}^\circ$, ν is the kinematic viscosity, α is the coefficient of thermal expansivity, g is the acceleration of gravity, and χ is the coefficient of temperature conductivity. For (22), the matrix λ has the form

$$\lambda = \begin{pmatrix} \nu k^2 & -\alpha g \dfrac{\kappa^2}{k^2} \\ -\beta & \chi k^2 \end{pmatrix}. \tag{23}$$

The one-time correlation functions of fluctuations are determined by the locally equilibrium distribution function of fluctuations,

$$f(\mathbf{v}, \theta) \propto \exp\left[-\int \left(\frac{\mathbf{v}^2}{2T} + \frac{c_V \theta^2}{2T^2}\right) \rho dV\right], \tag{24}$$

where ρ is the liquid density, $\mathbf{v}$ is the fluctuation velocity, which consists of horizontal and vertical w components, and c_V is the specific heat at a constant volume. As follows from (24), there are no correlations between fluctuations of the vertical velocity and temperature in the problem. So, the matrix β^{-1} of the Fourier components of the vertical velocity and temperature takes the form

$$\beta^{-1} = \begin{pmatrix} \langle w_{\mathbf{k}} w_{\mathbf{k}'} \rangle & 0 \\ 0 & \langle \theta_{\mathbf{k}} \theta_{\mathbf{k}'} \rangle \end{pmatrix}. \tag{25}$$

The matrix β^{-1} and kinetic coefficients for the Rayleigh–Bénard problem were obtained in [14, 15]. The asymmetry of kinetic coefficients is visible even without explicit expressions for $\langle w_{\mathbf{k}} w_{\mathbf{k}'} \rangle$ and $\langle \theta_{\mathbf{k}} \theta_{\mathbf{k}'} \rangle$:

$$\gamma = \begin{pmatrix} \nu k^2 \langle w_{\mathbf{k}} w_{\mathbf{k}'} \rangle & -\alpha g \dfrac{\kappa^2}{k^2} \langle \theta_{\mathbf{k}} \theta_{\mathbf{k}'} \rangle \\ -\beta \langle w_{\mathbf{k}} w_{\mathbf{k}'} \rangle & \chi k^2 \langle \theta_{\mathbf{k}} \theta_{\mathbf{k}'} \rangle \end{pmatrix}. \tag{26}$$

From (22) and (25), we now get the cross-correlation functions

$$\langle w_{\mathbf{k}}(t) \theta_{\mathbf{k}'} \rangle = \langle \theta_{\mathbf{k}} \theta_{\mathbf{k}'} \rangle \frac{\alpha g \kappa^2 / k^2}{\Omega_- - \Omega_+} (e^{-\Omega_+ t} - e^{-\Omega_- t}), \tag{27}$$

$$\langle \theta_{\mathbf{k}}(t) w_{\mathbf{k}'} \rangle = \langle w_{\mathbf{k}} w_{\mathbf{k}'} \rangle \frac{\beta}{\Omega_- - \Omega_+} (e^{-\Omega_+ t} - e^{-\Omega_- t}), \tag{28}$$

where

$$\Omega_\pm = \frac{1}{2}\left[(\nu + \lambda)k^2 \pm \sqrt{(\nu - \chi)^2 k^4 + 4\alpha\beta g \frac{\kappa^2}{k^2}}\right]. \tag{29}$$

Onsager's reciprocal relations are not satisfied, since $\langle w_{\mathbf{k}}(t) \theta_{\mathbf{k}'} \rangle \neq -\langle \theta_{\mathbf{k}'}(t) w_{\mathbf{k}} \rangle$; while, for the cross-sectional kinetic coefficients (26) and the correlation functions (27) and (28), (13) takes place.

Similarly, generalizations of the above results can be found in the case of a greater number of equations (3). However, the additional conditions for the coefficients λ arise. So, in the case of three equations, for the fulfillment of the three relations of reciprocity (10), the condition $\lambda_{12}\lambda_{23}\lambda_{31} = \lambda_{13}\lambda_{32}\lambda_{21}$ is required. Moreover, it is also assumed that all $\lambda_{ij}(i \neq j)$ are nonzero.

5. Conclusions

All features of the behavior of hydrodynamic fluctuations can be obtained on the basis of Onsager's regression hypothesis and a local equilibrium in hydrodynamics. In the same framework, the properties of the time symmetry of the correlation functions of hydrodynamic fluctuations can be elucidated. In OHSS, the reciprocity relations turn out to be different from Onsager's reciprocal relations, which occurs for fluctuations near the equilibrium state.

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СПІВВІДНОШЕННЯ ВЗАЄМНОСТІ
ДЛЯ ВІДКРИТИХ ГІДРОДИНАМІЧНИХ
СТАЦІОНАРНИХ СИСТЕМ (ВГСС)

Р е з ю м е

Виходячи з регресивної гіпотези Онзагера і локальної рівноваги в гідродинаміці, симетрія за часом взаємних кореляційних функцій флуктуацій аналізується безпосередньо з макроскопічних рівнянь руху поблизу деякого стаціонарного стану в суцільному середовищі. Показано, що в ВГСС потік порушує симетрію кореляційних функцій та співвідношення взаємності Онзагера, які мають місце поблизу рівноважного стаціонарного стану. Знайдено співвідношення взаємності в ВГСС. Розглянуто приклади їх використання.