SHARP WEAK TYPE ESTIMATES FOR MAXIMAL OPERATORS ASSOCIATED TO RARE BASES

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Dedicated to Dmitriy Dmitrishin on the occasion of his fifty-fifth birthday

Abstract. Let $\mathcal{B}$ denote a nonempty translation invariant collection of intervals in $\mathbb{R}^n$ (which we regard as a rare basis), and define the associated geometric maximal operator $M_{\mathcal{B}}$ by

$$M_{\mathcal{B}}f(x) = \sup_{x\in R\in\mathcal{B}} \frac{1}{|R|}\int_R |f|.$$

We provide a sufficient condition on $\mathcal{B}$ so that the estimate

$$|\{x \in \mathbb{R}^n : M_{\mathcal{B}}f(x) > \alpha\}| \leq C_n \int_{\mathbb{R}^n} \frac{|f|}{\alpha}\left(1 + \log^+ \frac{|f|}{\alpha}\right)^{n-1}$$

is sharp. As a corollary we obtain sharp weak type estimates for maximal operators associated to several classes of rare bases including Córdoba, Soria and Zygmund bases.

1. Introduction

Let $\mathcal{B}$ be a *basis* in $\mathbb{R}^n$, which we can treat as a collection of sets of positive finite measure covering $\mathbb{R}^n$. We may associate to $\mathcal{B}$ a *geometric maximal operator* $M_{\mathcal{B}}$ defined on measurable functions f on $\mathbb{R}^n$ by

$$M_{\mathcal{B}}f(x) = \sup_{x\in R\in\mathcal{B}} \frac{1}{|R|}\int_R |f| \ .$$

Two well-known examples of geometric maximal operators include the *Hardy-Littlewood maximal operator* M_{HL} and the *strong maximal operator* M_S. For the Hardy-Littlewood maximal operator, the basis $\mathcal{B}$ consists of all cubic intervals in $\mathbb{R}^n$; for the strong maximal operator, the basis consists of all intervals in $\mathbb{R}^n$. (For clarity, in this paper an *interval* in $\mathbb{R}^n$ is a rectangular parallelepiped whose sides are parallel to the coordinate axes.)

The Hardy-Littlewood maximal operator satisfies the weak type $(1, 1)$ estimate

$$|\{x \in \mathbb{R}^n : M_{HL}f(x) > \alpha\}| \leq C_n \int_{\mathbb{R}^n} \frac{|f|}{\alpha}.$$

2020 *Mathematics Subject Classification.* Primary 42B25.
Key words and phrases. maximal operator, weak type estimate, basis.
P. H. is partially supported by a grant from the Simons Foundation (#521719 to Paul Hagelstein).

The strong maximal operator satisfies the weaker estimate

$$|\{x \in \mathbb{R}^n : M_S f(x) > \alpha\}| \leq C_n \int_{\mathbb{R}^n} \frac{|f|}{\alpha}\left(1 + \log^+ \frac{|f|}{\alpha}\right)^{n-1} \tag{1}$$

and moreover this inequality is *sharp* in the sense that, if $\phi : [0,\infty) \to [0,\infty)$ is a convex increasing function with $\lim_{u\to\infty} \frac{\phi(u)}{u(1+\log u)^{n-1}} = 0$, then there is no finite constant $C_{n,\phi}$ such that the estimate

$$|\{x \in \mathbb{R}^n : M_S f(x) > \alpha\}| \leq C_{n,\phi} \int_{\mathbb{R}^n} \phi\left(\frac{|f|}{\alpha}\right)$$

holds for all measurable functions f and $\alpha > 0$.

Geometric maximal operators associated to *rare bases* occupy a fascinating middle ground between the Hardy-Littlewood and strong maximal operators. Significant mathematical work on the topic on rare bases has been done by, among others, Zygmund [14], Córdoba [1], Soria [9], and Rey [8]. For the purposes of this paper, a rare basis is a translation invariant collection of some (but not all) intervals in $\mathbb{R}^n$. The problem of finding a sharp weak type $\phi(L)$ estimate for the maximal operator $M_{\mathcal{B}}$ associated to a rare basis $\mathcal{B}$ is of central importence in the area. A natural conjecture (see, e.g., [5]) is $M_{\mathcal{B}}$ must satisfy a sharp weak type $L(1+\log^+ L)^k$ estimate for some $k \in \{0, 1, \ldots, n-1\}$. This conjecture in the two-dimensional case was proven by Stokolos [10] (see also [13]). Note that for any rare basis $\mathcal{B}$, the associated geometric maximal operator $M_{\mathcal{B}}$ is dominated by the strong maximal operator M_S and hence $M_{\mathcal{B}}$ automatically satisfies the weak type $L(1 + \log^+ L)^{n-1}$ estimate (1). Results of Stokolos [10, 11]; Dmitrishin, Hagelstein, and Stokolos [3, 5, 6] and D'Aniello and Moonens [2] have provided examples of a variety of multi-dimensional rare bases in which this worst-case estimate is sharp. The purpose of this paper is to present a generalization of results in these papers that provides a set of conditions on a rare basis $\mathcal{B}$ that will guarantee the sharpness of the weak type $L(1 + \log^+ L)^{n-1}$ estimate on $M_{\mathcal{B}}$.

In Section 2 we will state the main theorem of this paper and provide the associated requisite terminology. Section 3 will be devoted to a proof of the theorem. In Section 4 we will provide a useful generalization of our main theorem and subsequently provide applications of our results, in particular showing how they imply that Córdoba, Soria and Zygmund bases in $\mathbb{R}^n$ generate geometric maximal operators for which the weak type $L(1+\log^+ L)^{n-1}$ estimate is sharp.

2. Terminology and Statement of Main Theorem

Let $\mathcal{B}$ be a rare basis in $\mathbb{R}^n$. The *spectrum* of $\mathcal{B}$ will be defined as the set of all n-tuples of the type

$$(\lceil \log |R_1| \rceil, \ldots, \lceil \log |R_n| \rceil)$$

where $R_1 \times \cdots \times R_n \in \mathcal{B}$, $\lceil x \rceil$ denotes the least integer greater than or equal to x and here and below log stands for $\log_2$. The spectrum will be denoted by $W_{\mathcal{B}}$.

Let us call a set $W \subset \mathbb{Z}$ a *net* for a set $S \subset \mathbb{Z}$ if there exists a number $N \in \mathbb{N}$ such that for every $s \in S$ there exists $w \in W$ with $|s - w| \le N$.

Given a set $W \subset \mathbb{Z}^n$ and $t \in \mathbb{Z}^{n-1}$, we let W_t denote the set $\{\tau \in \mathbb{Z} : (t, \tau) \in W\}$.

Let us call a set $W \subset \mathbb{Z}^n$ a *net* for a set $S \subset \mathbb{Z}^n$ if W_t is a net for S_t for every $t \in \mathbb{Z}^{n-1}$.

If $k \in \{1, 2, \dots, n\}$, we let $\pi_k : \mathbb{R}^n \to \mathbb{R}^k$ denote the projection defined by $\pi_k(x_1, \dots, x_n) = (x_1, \dots, x_k)$.

If $S_1, \dots, S_n \subset \mathbb{Z}$, we say that the set $W \subset \mathbb{Z}^n$ is *dense* in the set $S_1 \times \dots \times S_n$ provided the sets

$$\pi_1(W),\ \pi_2(W),\ \dots,\ \pi_n(W) = W$$

are nets for the sets

$$S_1,\ \pi_1(W) \times S_2,\ ...,\ \pi_{n-1}(W) \times S_n,$$

respectively.

Our main theorem is the following.

Theorem 1. *If for a rare basis $\mathcal{B}$ in $\mathbb{R}^n$ there exist infinite sets $S_1, \dots, S_n \subset \mathbb{Z}$ for which the spectrum of $\mathcal{B}$ is dense in $S_1 \times \dots \times S_n$, then the maximal operator $M_{\mathcal{B}}$ satisfies a sharp weak type $L(1 + \log^+ L)^{n-1}$ estimate. Moreover, for every $\alpha \in (0, 1)$ there exists a bounded set $E_\alpha \subset \mathbb{R}^n$ with positive measure such that*

$$|\{x \in \mathbb{R}^n : M_{\mathcal{B}}(\chi_{E_\alpha})(x) > \alpha\}| \ge c_n \frac{1}{\alpha}\left(1 + \log \frac{1}{\alpha}\right)^{n-1} |E_\alpha|.$$

3. Proof of Theorem 1

Lemma 1. *Let $\mathcal{B}$ be a rare basis and suppose each of the intervals from $\mathcal{B}$ has dyadic side lengths. Suppose $S_1, \dots, S_n \subset \mathbb{Z}$ are infinite sets and the spectrum of $\mathcal{B}$ is dense in $S_1 \times \dots \times S_n$. Then for every $k \in \mathbb{N}$ there exist increasing sequences $(s_{1,m})_{m=0}^k, \dots, (s_{n,m})_{m=0}^k$ with members from $S_1, \dots, S_n$ respectively such that for every n-tuple $(m_1, \dots, m_n)$ belonging to $\{1, \dots, k\}^n$ there exists an interval $R_1 \times \dots \times R_n$ from the basis $\mathcal{B}$ for which $|R_1| \in (2^{s_{1,m_1-1}}, 2^{s_{1,m_1}}], \dots, |R_n| \in (2^{s_{n,m_n-1}}, 2^{s_{n,m_n}}]$.*

Proof. Since $\pi_1(W_{\mathcal{B}})$ is a net for S_1, there exists a number N such that for every $s \in S_1$ there exists $\tau \in \pi_1(W_{\mathcal{B}})$ with $|s - \tau| \le N$. Let $\alpha_{1,0} < \dots < \alpha_{1,2k}$ be numbers from S_1 such that $\alpha_{1,m} - \alpha_{1,m-1} > N$ for every $m \in \{1, \dots, 2k\}$. Set $s_{1,m} = \alpha_{1,2m}$ $(m \in \{0, \dots, k\})$. Then it is easy to see that

$$\pi_1(W_{\mathcal{B}}) \cap (s_{1,m-1}, s_{1,m}] \ne \emptyset \tag{2}$$

for every $m \in \{1, \dots, k\}$.

Suppose that for some $j < n$ increasing sequences $(s_{1,m})_{m=0}^k, \dots, (s_{j,m})_{m=0}^k$ with members from $S_1, \dots, S_j$ respectively are constructed.

Let us consider an arbitrary $(t_1, \dots, t_j) \in \pi_j(W_{\mathcal{B}})$ such that $t_1 \in [s_{1,0}, s_{1,k}], \dots, t_j \in [s_{j,0}, s_{j,k}]$. Let

$$W_{\mathcal{B},t_1,\dots,t_j} = \{\tau \in \mathbb{Z} : (t_1, \dots, t_j, \tau) \in \pi_{j+1}(W_{\mathcal{B}})\}.$$

Since $W_{\mathcal{B}}$ is dense in $S_1 \times \cdots \times S_n$ we have $W_{\mathcal{B},t_1,\ldots,t_j}$ is a net for S_{j+1}. Let $N_{t_1,\ldots,t_j}$ be the number such that for every $s \in S_{j+1}$ there exists $\tau \in W_{\mathcal{B},t_1,\ldots,t_j}$ for which $|s-\tau| \le N_{t_1,\ldots,t_j}$.

Denote by N the largest of the numbers $N_{t_1,\ldots,t_j}$ where $(t_1,\ldots,t_j) \in \pi_j(W_{\mathcal{B}})$ and $t_1 \in [s_{1,0}, s_{1,k}]$, $\ldots$, $t_j \in [s_{j,0}, s_{j,k}]$. Let $\alpha_{j+1,0} < \cdots < \alpha_{j+1,2k}$ be numbers from S_{j+1} such that $\alpha_{j+1,m} - \alpha_{j+1,m-1} > N$ for every $m \in \{1,\ldots,2k\}$. Set $s_{j+1,m} = \alpha_{j+1,2m}$ $(m \in \{0,\ldots,k\})$. Then for every $(t_1,\ldots,t_j) \in \pi_j(W_{\mathcal{B}})$ with $t_1 \in [s_{1,0}, s_{1,k}], \ldots, t_j \in [s_{j,0}, s_{j,k}]$ we have that

$$W_{\mathcal{B},t_1,\ldots,t_j} \cap (s_{j+1,m-1}, s_{j+1,m}] \ne \emptyset \tag{3}$$

for every $m \in \{1,\ldots,k\}$.

Taking into account (2) and (3), it is easy to check that the sequences $(s_{1,m})_{m=0}^k, \ldots, (s_{n,m})_{m=0}^k$ constructed in such a way have the needed property. The lemma is proved. □

Suppose $E \subset \mathbb{R}$ is a measurable set, Ω is a collection of disjoint closed one-dimensional intervals, $H = \bigcup_{I\in\Omega} I$ and $0 \le \alpha \le 1$. We will say that E *α-saturates* H (notation: $E \leftarrow^{\alpha} H$) if $|I \cap E|/|I| = \alpha$ for every $I \in \Omega$. For the case $\alpha = 1/2$ we will write simply $E \leftarrow H$.

Note that:

a) If $E \leftarrow^{\alpha} H$ then $|H \cap E|/|H| = \alpha$;

b) If $E \subset H$, $E \leftarrow^{\alpha} H$ and $H \leftarrow^{\beta} T$ then $E \leftarrow^{\alpha\beta} T$.

Let I and J be closed one-dimensional intervals with lengths 2^p and 2^q where $p, q \in \mathbb{Z}$ and $p < q$. Let us consider the partition of J into non-overlapping closed subintervals $J_1, \ldots, J_{2^{p-q}}$ having the same length as I and such that $\min J_1 < \cdots < \min J_{2^{p-q}}$. By $\langle I, J\rangle$ denote the union of the intervals J_k with odd indices. Obviously, $\langle I, J\rangle \subset J$ and $\langle I, J\rangle \leftarrow J$.

Let I be a closed one-dimensional interval with a length 2^p where $p \in \mathbb{Z}$, Ω be a collection of disjoint closed one-dimensional intervals having a length 2^q with $q \in \mathbb{Z}$ and $p < q$, and $H = \bigcup_{J\in\Omega} J$. By $\langle I, H\rangle$ we denote the union $\bigcup_{J\in\Omega} \langle I, J\rangle$. Obviously, $\langle I, H\rangle \subset H$ and $\langle I, H\rangle \leftarrow H$.

It is easy to check the validity of the following statement.

Lemma 2. *Suppose $s_0 < s_1 < \cdots < s_k$ are some integers, $I_0 = [0, 2^{s_0}], \ldots, I_k = [0, 2^{s_k}]$ and*

$$I_k^* = I_k,\ I_{k-1}^* = \langle I_{k-1}, I_k^*\rangle,\ I_{k-2}^* = \langle I_{k-2}, I_{k-1}^*\rangle,\ \ldots,\ I_0^* = \langle I_0, I_1^*\rangle.$$

Then the sets $I_0^*, I_1^*, \ldots, I_k^*$ have the following properties:

1) $I_0^* \subset I_1^* \subset \cdots \subset I_k^*$;

2) $I_0^* \leftarrow I_1^* \leftarrow \cdots \leftarrow I_k^*$ and, moreover, $I_{m-1}^* \leftarrow J$ for every $m \in \{1,\ldots,k\}$ and every dyadic interval J contained in I_m^* whose length belongs to $(2^{s_{m-1}}, 2^{s_m}]$;

3) $I_0^* \leftarrow^{1/2^m} I_m^*$ for every $m \in \{1,\ldots,k\}$ and, moreover, $I_0^* \leftarrow^{1/2^m} J$ for every $m \in \{1,\ldots,k\}$ and every dyadic interval J contained in I_m^* whose length belongs to $(2^{s_{m-1}}, 2^{s_m}]$.

Remark 1. It is important to recognize that the interval J in 3) above may have not only length 2^{s_m} but also the “intermediate” lengths $2^{s_{m-1}+1}, \ldots, 2^{s_m-1}$ as well.

Remark 2. The idea of using the sets $I_0^*, I_1^*, \ldots, I_k^*$ in Lemma 2 for the study of weak type estimates for maximal operators associated to rare bases goes back to Stokolos [11].

For every $k \geq n$, by $\Omega_{n,k}$ we denote the set of all n-tuples $(m_1, \dots, m_n) \in \mathbb{N}^n$ for which $m_1 + \cdots + m_n = k$. Clearly, $\operatorname{card}(\Omega_{n,k}) \leq k^{n-1}$. On the other hand, it is easy to see that $\operatorname{card}(\Omega_{n,k}) \geq c_n k^{n-1}$.

Lemma 3. *For every $k \geq n$ and increasing sequences of integers*

$$(s_{1,m})_{m=0}^k, \ \dots \ , (s_{n,m})_{m=0}^k$$

there exists a bounded set $E \subset \mathbb{R}^n$ of positive measure with the following property: if $\mathcal{B}$ is a rare basis in $\mathbb{R}^n$ and Ω is the set of all n-tuples $(m_1, \dots, m_n) \in \Omega_{n,k}$ for which there exists an interval $R_1 \times \cdots \times R_n \in B$ with dyadic side lengths such that

$$|R_1| \in (2^{s_{1,m_1-1}}, 2^{s_{1,m_1}}], \dots, |R_n| \in (2^{s_{n,m_n-1}}, 2^{s_{n,m_n}}],$$

then

$$|\{x \in \mathbb{R}^n : M_{\mathcal{B}}(\chi_E)(x) \geq 1/2^k\}| \geq c_n \operatorname{card}(\Omega)\, 2^k |E|.$$

Proof. For every $j \in \{1, \dots, n\}$ denote by $I_{j,0}, \dots, I_{j,k}$ and $I^*_{j,0}, \dots, I^*_{j,k}$ the sets corresponding to the sequence $s_{j,0}, \dots, s_{j,k}$ according to Lemma 2. We define the set $E \subset \mathbb{R}^n$ by

$$E = I^*_{1,0} \times \cdots \times I^*_{n,0} \ .$$

We now show that

$$\{x \in \mathbb{R}^n : M_{\mathcal{B}}(\chi_E)(x) \geq 1/2^k\} \supset \bigcup_{(m_1,\dots,m_n)\in\Omega} I^*_{1,m_1} \times \cdots \times I^*_{n,m_n}. \tag{4}$$

Let us consider an arbitrary $(m_1, \dots, m_n) \in \Omega$. Let $R_1 \times \cdots \times R_n$ be an interval from $\mathcal{B}$ such that $|R_1|, \dots, |R_n|$ are dyadic numbers and $|R_j| \in (2^{s_{j,m_j-1}}, 2^{s_{j,m_j}}]$ for every $j \in \{1, \dots, n\}$. Then for every $j \in \{1, \dots, n\}$ each component interval of I^*_{j,m_j} can be decomposed into pairwise non-overlapping subintervals Δ each of whose lengths is equal to $|R_j|$. By Lemma 2 (see statement 3)) each Δ will be $1/2^{m_j}$-saturated by the set $I^*_{j,0}$. Hence, we can decompose the set $I^*_{1,m_1} \times \cdots \times I^*_{n,m_n}$ into pairwise non-overlapping intervals $\Delta_1 \times \cdots \times \Delta_n$ each of which is a translate of $R_1 \times \cdots \times R_n$ and

$$\frac{|(\Delta_1 \times \cdots \times \Delta_n) \cap E|}{|\Delta_1 \times \cdots \times \Delta_n|} = \frac{|(\Delta_1 \times \cdots \times \Delta_n) \cap (I^*_{1,0} \times \cdots \times I^*_{n,0})|}{|\Delta_1 \times \cdots \times \Delta_n|} =$$

$$\frac{|\Delta_1 \cap I^*_{1,0}|}{|\Delta_1|} \cdots \frac{|\Delta_n \cap I^*_{n,0}|}{|\Delta_n|} = \frac{1}{2^{m_1}} \cdots \frac{1}{2^{m_n}} = \frac{1}{2^k}.$$

Hence, $\{x \in \mathbb{R}^n : M_{\mathcal{B}}(\chi_E)(x) \geq 1/2^k\} \supset I^*_{1,m_1} \times \cdots \times I^*_{n,m_n}$. Consequently, taking into account that $(m_1, \dots, m_n)$ is arbitrary in Ω, we conclude (4) holds.

For any $j \in \{1, \dots, n\}$ we denote $H_{j,0} = I^*_{j,0}$ and $H_{j,m} = I^*_{j,m} \setminus I^*_{j,m-1}$ for $m \in \{1, \dots, k\}$. By virtue of Lemma 2 it is easy to see that:

i) The sets $H_{1,m_1} \times \cdots \times H_{n,m_n}$ $((m_1,\ldots,m_n) \in \Omega_{n,k})$ are pairwise disjoint;

ii) For every $j \in \{1,\ldots,n\}$ and $m \in \{1,\ldots,k\}$ we have that $|H_{j,m}| = |I^*_{j,m}|/2$. Consequently, for every $(m_1,\ldots,m_n) \in \Omega_{n,k}$,

$$|H_{1,m_1} \times \cdots \times H_{n,m_n}| = \frac{1}{2^n}|I^*_{1,m_1} \times \cdots \times I^*_{n,m_n}|;$$

iii) For every $(m_1,\ldots,m_n) \in \Omega_{n,k}$,

$$|I^*_{1,m_1} \times \cdots \times I^*_{n,m_n}| = 2^k|I^*_{1,0} \times \cdots \times I^*_{n,0}|.$$

Hence,

$$\begin{aligned}\Bigg|\bigcup_{(m_1,\ldots,m_n)\in\Omega} & I^*_{1,m_1} \times \cdots \times I^*_{n,m_n}\Bigg| \\ &\geq \Bigg|\bigcup_{(m_1,\ldots,m_n)\in\Omega} H_{1,m_1} \times \cdots \times H_{n,m_n}\Bigg| \\ &= \sum_{(m_1,\ldots,m_n)\in\Omega} |H_{1,m_1} \times \cdots \times H_{n,m_n}| \\ &= \text{card}\,(\Omega)\frac{1}{2^n}|I^*_{1,0} \times \cdots \times I^*_{n,0}| \\ &= \frac{1}{2^n}\text{card}\,(\Omega)2^k|E|. \qquad (5)\end{aligned}$$

From (4) and (5) we conclude the lemma holds. □

For an interval R we denote by R_d the smallest interval concentric to R which contains R and has dyadic side lengths.

Let $\mathcal{B}$ be a rare basis in $\mathbb{R}^n$. To $\mathcal{B}$ we can associate its *dyadic skeleton* $\mathcal{B}_d = \{R_d : R \in \mathcal{B}\}$. Note that the maximal operators associated with the bases $\mathcal{B}$ and $\mathcal{B}_d$ possess similar properties. Namely, $M_{\mathcal{B}}f \leq 2^n M_{\mathcal{B}_d}f$ and on the other hand (see, e.g., [7], Lemma 2.12),

$$|\{x \in \mathbb{R}^n : M_{\mathcal{B}_d}f(x) > \alpha\}| \leq C_n|\{x \in \mathbb{R}^n : M_{\mathcal{B}}f(x) > \alpha/4^n\}|. \qquad (6)$$

Theorem 1 follows from Lemmas 1 and 3 and estimate (6).

4. A Generalization of Theorem 1 and Applications

Let $k \geq n$ and $\Omega \subset \Omega_{n,k}$. We will say that a rare basis $\mathcal{B}$ in $\mathbb{R}^n$ is Ω-*complete* if there exist increasing sequences of integers $(s_{1,m})_{m=0}^k, \ldots, (s_{n,m})_{m=0}^k$ such that for every n-tuple $(m_1,\ldots,m_n)$ belonging to Ω there exists an interval $R_1 \times \cdots \times R_n \in \mathcal{B}_d$ with $|R_1| \in (2^{s_{1,m_1-1}}, 2^{s_{1,m_1}}], \ldots, |R_n| \in (2^{s_{n,m_n-1}}, 2^{s_{n,m_n}}]$.

From Lemma 3 and estimate (6) we obtain the following result.

Theorem 2. *Let $\mathcal{B}$ be a rare basis in $\mathbb{R}^n$. Suppose there exist an increasing sequence of natural numbers $k_j \geq n$ and a sequence of sets $\Omega_j \subset \Omega_{n,k_j}$ with the properties: $\mathcal{B}$ is Ω_j-complete for every $j \in \mathbb{N}$ and $\inf_{j\in\mathbb{N}} \operatorname{card}(\Omega_j)/k_j^{n-1} > 0$. Then $M_{\mathcal{B}}$ satisfies the sharp weak type $L(1+\log^+ L)^{n-1}$ estimate. Moreover, for every $j \in \mathbb{N}$ there exists a bounded set $E_j \subset \mathbb{R}^n$ with positive measure such that*

$$|\{x \in \mathbb{R}^n : M_{\mathcal{B}}(\chi_{E_j})(x) \geq 1/2^{k_j}\}| \geq c_{\mathcal{B}} k_j^{n-1} 2^{k_j} |E_j|,$$

where $c_{\mathcal{B}}$ is a constant of the form $c_n \inf_{j\in\mathbb{N}} \operatorname{card}(\Omega_j)/k_j^{n-1}$.

Theorem 2 is an extension of Theorem 1 since by Lemma 1 the density of the spectrum $W_{\mathcal{B}}$ in the Cartesian product of infinite sets $S_1, \dots, S_n \subset \mathbb{Z}$ implies $\Omega_{n,k}$-completeness of the basis $\mathcal{B}$ for every $k \geq n$.

We now indicate eight applications of Theorems 1 and 2 to rare bases.

I. Let $S_1, \dots, S_n \subset \mathbb{Z}$ be infinite sets and $\mathcal{B}$ be the basis consisting of all n-dimensional intervals with side lengths of the form $2^{s_1}, \dots, 2^{s_n}$ where $s_1, \dots, s_n$ belong to the sets $S_1, \dots, S_n$ respectively. Taking into account that the spectrum of $\mathcal{B}$ is the product $S_1 \times \cdots \times S_n$ and applying Theorem 1 for $\mathcal{B}$ and $S_1, \dots, S_n$ we obtain the result proved in [11].

II (Soria Bases). Let $\Gamma \subset \mathbb{Z}$ be an infinite set and let $\mathcal{B}$ be the basis of all 3-dimensional intervals $R_1 \times R_2 \times R_3$ such that $|R_1|, |R_2| \in \mathbb{D}$ and $|R_3| = 2^\gamma/|R_2|$ for some $\gamma \in \Gamma$, where here and in later applications we denote the set of dyadic numbers $\{2^s : s \in \mathbb{Z}\}$ by $\mathbb{D}$. It is easy to see that the spectrum $W_{\mathcal{B}}$ is the set

$$\{(w_1, w_2, w_3) : w_1, w_2 \in \mathbb{Z}, w_3 \in \Gamma - w_2\}$$

and $W_{\mathcal{B}}$ is dense in $\mathbb{Z} \times \mathbb{Z} \times \Gamma$. Hence taking $S_1 = S_2 = \mathbb{Z}$ and $S_3 = \Gamma$ by Theorem 1 we obtain the sharp weak type $L(1+\log^+ L)^2$ estimate for the maximal operator $M_{\mathcal{B}}$ associated to the basis $\mathcal{B}$ which was proved in [3].

III (Zygmund Bases). Let $\Gamma \subset \mathbb{Z}$ be an infinite set and let $\mathcal{B}$ be the basis of all 3-dimensional intervals $R_1 \times R_2 \times R_3$ such that $|R_1|, |R_2| \in \mathbb{D}$ and $|R_3| = 2^\gamma |R_2|$ for some $\gamma \in \Gamma$. It is easy to see that the spectrum $W_{\mathcal{B}}$ is the set

$$\{(w_1, w_2, w_3) : w_1, w_2 \in \mathbb{Z}, w_3 \in \Gamma + w_2\}$$

and $W_{\mathcal{B}}$ is dense in $\mathbb{Z} \times \mathbb{Z} \times \Gamma$. Hence taking $S_1 = S_2 = \mathbb{Z}$ and $S_3 = \Gamma$ by Theorem 1 we obtain the sharp weak type $L(1+\log^+ L)^2$ estimate for the maximal operator $M_{\mathcal{B}}$ associated to the basis $\mathcal{B}$ which was proved in [5].

IV (Córdoba Bases). Let $\Gamma \subset \mathbb{Z}$ be an infinite set and let $\mathcal{B}$ be the basis of all 3-dimensional intervals $R_1 \times R_2 \times R_3$ such that $|R_1|, |R_2| \in \mathbb{D}$ and $|R_3| = 2^\gamma |R_1||R_2|$ for some $\gamma \in \Gamma$. It is easy to see that the spectrum $W_{\mathcal{B}}$ is the set

$$\{(w_1, w_2, w_3) : w_1, w_2 \in \mathbb{Z}, w_3 \in \Gamma + w_1 + w_2\}.$$

and $W_{\mathcal{B}}$ is dense in $\mathbb{Z}\times\mathbb{Z}\times\Gamma$. Hence, taking $S_1 = S_2 = \mathbb{Z}$ and $S_3 = \Gamma$ by Theorem 1 we obtain the sharp weak type $L(1+\log^+ L)^2$ estimate for the maximal operator $M_{\mathcal{B}}$ associated to the basis $\mathcal{B}$ which was proved in [6].

V. Suppose $T_1,\dots,T_{n-1}$ are infinite subsets of $\mathbb{D}$, Γ is an infinite subset of $\mathbb{Z}$, $1\le p\le n-1$, and $1\le j_1<\dots<j_p\le n-1$. Let $\mathcal{B}$ be the basis of all n-dimensional intervals $R_1\times\dots\times R_n$ such that $|R_1|\in T_1,\dots,|R_{n-1}|\in T_{n-1}$ and $|R_n| = 2^\gamma/(|R_{j_1}|\dots|R_{j_p}|)$ for some $\gamma\in\Gamma$.

Let $S_j = \{\log k : k\in T_j\}$ $(j\in\{1,\dots,n-1\})$ and $S_n=\Gamma$.

It is easy to see that the spectrum $W_{\mathcal{B}}$ is the set

$$\{(w_1,\dots,w_{n-1},w_n) : w_1\in S_1,\dots,w_{n-1}\in S_{n-1}, w_n\in\Gamma-(w_{j_1}+\dots+w_{j_p})\}$$

and $W_{\mathcal{B}}$ is dense in $S_1\times\dots\times S_n$.

Applying Theorem 1 for $\mathcal{B}$ and $S_1,\dots,S_n$ we obtain the sharp weak type $L(1+\log^+ L)^{n-1}$ estimate for the maximal operator $M_{\mathcal{B}}$ associated to the basis $\mathcal{B}$.

Under the same conditions we can obtain the sharp weak type $L(1+\log^+ L)^{n-1}$ estimate for the maximal operator $M_{\mathcal{B}}$ associated to the basis $\mathcal{B}$ of all intervals $R_1\times\dots\times R_n$ such that $|R_1|\in T_1,\dots,|R_{n-1}|\in T_{n-1}$ and $|R_n| = 2^\gamma|R_{j_1}|\dots|R_{j_p}|$ for some $\gamma\in\Gamma$.

Note that the bases considered in this application are multi-dimensional versions of ones from applications II-IV.

VI. The conditions of Theorem 1 are satisfied by more general bases than ones considered in the applications II-V. In particular, let $\theta_k:\mathbb{D}^{n-1}\to\mathbb{D}$ $(k\in\mathbb{N})$ be functions satisfying the following conditions:

1) $\inf\limits_{k\in\mathbb{N}}\theta_k(1,\dots,1)=0$ or $\sup\limits_{k\in\mathbb{N}}\theta_k(1,\dots,1)=\infty$;

2) for every $(t_1,\dots,t_{n-1})\in\mathbb{D}^{n-1}$

$$\inf_{k\in\mathbb{N}}\frac{\theta_k(t_1,\dots,t_{n-1})}{\theta_k(1,\dots,1)}>0 \ \text{ and } \ \sup_{k\in\mathbb{N}}\frac{\theta_k(t_1,\dots,t_{n-1})}{\theta_k(1,\dots,1)}<\infty.$$

Suppose $T_1,\dots,T_{n-1}$ are infinite subsets of $\mathbb{D}$. Let $\mathcal{B}$ be the basis of all n-dimensional intervals $R_1\times\dots\times R_n$ such that $|R_1|\in T_1,\dots,|R_{n-1}|\in T_{n-1}$ and $|R_n|=\theta_k(|R_1|,\dots,|R_{n-1}|)$ for some $k\in\mathbb{N}$.

Let $S_j=\{\log k : k\in T_j\}$ $(j\in\{1,\dots,n-1\})$ and $S_n=\{\log\theta_k(1,\dots,1) : k\in\mathbb{N}\}$.

It is easy to see that the spectrum $W_{\mathcal{B}}$ is the following set

$$\{(w_1,\dots,w_{n-1},\log\theta_k(2^{w_1},\dots,2^{w_{n-1}})) : w_1\in S_1,\dots,w_{n-1}\in S_{n-1}, k\in\mathbb{N}\}$$

and $W_{\mathcal{B}}$ is dense in $S_1\times\dots\times S_n$.

Applying Theorem 1 for $\mathcal{B}$ and $S_1,\dots,S_n$ we obtain the sharp weak type $L(1+\log^+ L)^{n-1}$ estimate for the maximal operator $M_{\mathcal{B}}$ associated to the basis $\mathcal{B}$.

VII. Following [12] let us say that a rare basis $\mathcal{B}$ in $\mathbb{R}^2$ satisfies the (is)-*property* if for every $k\in\mathbb{N}$ there exist intervals $R_0,\dots,R_k\in\mathcal{B}$ of the type $[0,2^p]\times[0,2^q]$ $(p,q\in\mathbb{Z})$ such that:

1) For every $i,j\in\{0,\dots,k\}$ with $i\ne j$ the intervals R_i and R_j are incomparable, i.e., there does not exist translation placing one of them inside the other;

2) For every $i, j \in \{0, \dots, k\}$ the interval $R_i \cap R_j$ belongs to $\mathcal{B}$.

Let $\mathcal{B}_1$ be a basis of two-dimensional intervals with the (is)-property and $\mathcal{B}_2$ be a basis consisting of one-dimensional intervals with lengths belonging to an infinite set of dyadic numbers. By $\mathcal{B}_1 \times \mathcal{B}_2$ denote their product, i.e., the basis which consists of three-dimensional intervals of the type $J_1 \times J_2$ where $J_1 \in \mathcal{B}_1$ and $J_2 \in \mathcal{B}_2$.

Let $k \geq 3$. We can find intervals $R_0, \dots, R_k \in \mathcal{B}_1$ of the type $[0, 2^p] \times [0, 2^q]$ $(p, q \in \mathbb{Z})$ with the properties 1) and 2) from the definition of the (is)-property. We can assume that

$$R_0 = [0, 2^{p_0}] \times [0, 2^{q_k}], \dots, R_i = [0, 2^{p_i}] \times [0, 2^{q_{k-i}}], \dots, R_k = [0, 2^{p_k}] \times [0, 2^{q_0}],$$

where $p_0 < \dots < p_k$ and $q_0 < \dots < q_k$. Let $t_0 < \dots < t_k$ be integers such that $[0, 2^{t_0}], \dots, [0, 2^{t_k}] \in \mathcal{B}_2$. Set $s_{1,0} = p_0, \dots, s_{1,k} = p_k$, $s_{2,0} = q_0, \dots, s_{2,k} = q_k$, and $s_{3,0} = t_0, \dots, s_{3,k} = t_k$. Then for every triple (m_1, m_2, m_3) belonging to $V_{3,k}$ it is easy to see that

$$[0, 2^{s_{1,m_1}}] \times [0, 2^{s_{2,m_2}}] \times [0, 2^{s_{3,m_3}}] = (R_{m_1} \cap R_{k-m_2}) \times [0, 2^{t_{m_3}}] \in \mathcal{B}_1 \times \mathcal{B}_2.$$

Hence, $\mathcal{B}_1 \times \mathcal{B}_2$ is $\Omega_{3,k}$-complete for every $k \geq 3$.

Applying Theorem 2 we obtain the sharp weak type $L(1+\log^+ L)^2$ estimate for the maximal operator $M_{\mathcal{B}}$ associated to the basis $\mathcal{B}$ where $\mathcal{B} = \mathcal{B}_1 \times \mathcal{B}_2$. For the case of $\mathcal{B}_2$ being the basis of all intervals with dyadic lengths the estimate was obtained in [12].

VIII. In Section 4.2 of [2] a certain class Λ_n of rare bases in $\mathbb{R}^n$ is considered such that every basis $\mathcal{B}$ from Λ_n has the following property (see Remark 9 in [2]): For every $k \in \mathbb{N}$ there exist intervals $R_0, R_1, \dots, R_k$ such that

$$\begin{aligned} R_0 &= [0, 2^{s_{1,0}}] \times [0, 2^{s_{2,0}}] \times \dots \times [0, 2^{s_{n,0}}], \\ R_1 &= [0, 2^{s_{1,1}}] \times [0, 2^{s_{2,1}}] \times \dots \times [0, 2^{s_{n,1}}], \\ &\vdots \\ R_k &= [0, 2^{s_{1,k}}] \times [0, 2^{s_{2,k}}] \times \dots \times [0, 2^{s_{n,k}}], \end{aligned}$$

where $(s_{1,m})_{m=0}^k, \dots, (s_{n,m})_{m=0}^k$ are increasing sequences of integers and for every n-tuple of integers $(m_1, \dots, m_n)$ with $k \geq m_1 \geq m_2 \geq \dots \geq m_n \geq 0$ the interval

$$R = [0, 2^{s_{1,m_1}}] \times [0, 2^{s_{2,m_2}}] \times \dots \times [0, 2^{s_{n,m_n}}]$$

belongs to the basis $\mathcal{B}$.

Suppose $\mathcal{B}_1$ is a basis from the class Λ_n and $\mathcal{B}_2$ is a basis consisting of one-dimensional intervals with lengths belonging to an infinite set of dyadic numbers. Taking into account the above given property of bases from the class Λ_n we have that the product basis $\mathcal{B}_1 \times \mathcal{B}_2$ is Ω_k-complete for every $k \geq n+1$ where Ω_k is the set of all $(n+1)$-tuples $(m_1, \dots, m_n, m_{n+1}) \in \Omega_{n+1,k}$ with $k \geq m_1 \geq m_2 \geq \dots \geq m_n \geq 1$. On the other hand, it is easy to see that $\operatorname{card} \Omega_k \geq c_n k^n$ $(k \geq n+1)$.

Applying Theorem 2 we obtain the sharp weak type $L(1+\log^+ L)^n$ estimate for the maximal operator $M_{\mathcal{B}}$ associated to the basis $\mathcal{B}$ where $\mathcal{B} = \mathcal{B}_1 \times \mathcal{B}_2$.

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