This paper has analyzed the equation of motion in terms of stresses (Navier), as well as its two special cases for an incompressible viscous current. One is the Stokes (Navier-Stokes) equation, and the other was derived with fewer restrictions. It has been shown that the Laplace equation of linear velocity can be represented as a function of two variables – the linear and angular speed of particle rotation.

To describe the particle acceleration, all motion equations employed a complete derivative from speed in the Gromeka-Lamb form, which depends on the same variables.

Taking into consideration the joint influence of linear and angular velocity allows solving a task of the analytical description of a turbulent current within the average model. A given method of analysis applies the provision of general physics that examines the translational and rotational motion. The third type of mechanical movement, oscillatory (pulsation), was not considered in the current work.

A property related to the Stokes equation decomposition has been found; a block diagram composed of equations and conditions has been built. It is shown that all equations for viscous liquid have their own analog in a simpler model of non-viscous fluid. That makes it easier to find solutions to the equations for the viscous flow.

The Stokes and Navier equations were used to solve two one-dimensional problems, which found the distribution of speed along the normal to the surface at the flow on a horizontal plate and in a circular pipe. Both solution methods produce the same result. No solution for the distribution of speed along the normal to the surface in a laminar sublayer could be found. A relevant task related to the mathematical part is to solve the problem of closing the equations considered.

A comparison of the theoretical and empirical equations has been performed, which has made it possible to justify the assumption that a rarefied gas is the Stokes liquid

Keywords: average turbulence model, viscous friction, Stokes equation, Navier equation

UDC 532.5.032

DOI: 10.15587/1729-4061.2021.225209

ANALYZING THE INFLUENCE OF A PARTICLE'S LINEAR AND ANGULAR VELOCITY ON THE EQUATIONS OF LIQUID MOTION

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Received date 29.12.2020
Accepted date 11.02.2021
Published date 26.02.2021

1. Introduction

Underlying the classic method of constructing mathematical models in fluid mechanics is the equation of motion in terms of stresses (Navier), which is a special case of the law of preserving the amount of movement [1–3]. The Navier equation has two special cases for viscous liquids such as the Stokes equation (Navier-Stokes) and the second one derived with fewer restrictions in work [4]. Both equations take into consideration the influence of mass forces, pressure forces, friction and inertia forces but have different expressions for two components – accelerations due to the forces of friction and pressure.

A characteristic feature of the Stokes equation is the lack of particle rotation effect while the second equation includes this influence [4]. Both equations are the special cases of the same equation (Navier), they employ Newton's rheological equation and should have the same functional dependence for the same components.

The noted contradiction requires clarification of the reasons for this discrepancy, which could improve the mathematical model of the current that is widely used in engineering practice.

The exact solutions to the Stokes equation are consistent with experiments only at low Reynolds numbers. There is a known Stokes solution for the movement of a ball in a Newtonian fluid, which is consistent with experimental data in experiments involving non-Newtonian fluid (glycerin or castor oil) [1, 2]. These contradictions between the theory and experiment have no satisfactory explanation.

Similar mathematical models are built and used in the theory of elasticity and thermal conductivity [5, 6]. They make it possible to calculate physical fields with high quality and at minimal experiment engagement. Computer programs used in fluid mechanics (Flowvision, Phoenics, etc.) produce good results only in a narrow range of changes in influencing factors while their solutions are often unstable (approximate). This flaw requires an experimental check of numerical calculations, increases the cost and timing of advancements [2, 3]. It is believed that one of the causes of these problems is the calculation equations themselves (Stokes, Reynolds, etc.).

One way to resolve existing issues is to take into consideration an additional influencing factor – the angular speed of particle rotation. A given property should play a key role in describing the flow process but it is not used in modern models [1–3].

Thus, it is a relevant task to search for the new forms of Stokes equation and exact solutions to them, derived according to the classical scheme in accordance with the provisions of general physics.

2. Literature review and problem statement

Within an average model, turbulence emerges when particles rotate and speed pulsations occur, otherwise there is a laminar mode. This physical model has been known for more than 100 years; however, the classical equations of motion (Stokes, Reynolds for the average turbulent current, equations of the boundary layer, etc.) do not take it into consideration, which contradicts the definition of turbulence and its key features [1–3].

Stokes equation is derived in two ways: using the general theorems of mathematics and applying the equation of motion in terms of stresses (Navier).

The largest number of studies report the analysis of the first derivation variant in order to obtain accurate and numerical solutions to various problems. This method of analysis is used in work [7]. Underlying numerical solutions is the Stokes equation for an incompressible liquid. It is shown that there is a large class of solutions for areas of different geometry. The process of solving is accompanied by the use of assumptions without clear physical meaning, there is no comparison with the experiment. This and other similar tasks relate to pure mathematics and prevent the resulting solutions from being used in practice.

An analysis of the second derivation technique made it possible to obtain two precise solutions for the flow in the pipe and on the horizontal plate in the form of common integrals. A special case of equation (1) was used to this end. These solutions could not be applied to practical tasks because their relationship with current modes is unknown. Most applications employ the following form of Stokes equation:

$$G-\frac{1}{\rho}\text{grad}p+\nu\cdot\nabla^2 u=\frac{du}{dt}. \tag{1}$$

It follows from (1) that the main factor taking into consideration the dynamics of the current is linear speed, which affects the components due to the forces of viscous friction and inertia. This is not consistent with the position of general mechanics where three types of movement are considered: translational, rotational, and oscillatory.

Paper [4] analyzes another special case of the Navier equation, which takes into consideration the angular velocity of the particle rotation.

This equation takes the following form:

$$G-\frac{1}{\rho}\text{div}p+2\nu\cdot f(u,\omega)=\frac{du}{dt}. \tag{2}$$

Both equations are derived under different limitations, given in Table 1.

In equation (1), the limitations refer to the tangent and normal stresses (τ, p_{xx}, p_{yy}, p_{zz}); in equation (2) – only to tangent ones. Thus, normal stresses (pressure) in equation (2) can change arbitrarily while in the Stokes equation – only according to the established rules, that is, the limitations are stricter. Ratios for the normal stresses (p_{xx}, p_{yy}, p_{zz}) make up the content of the linearity hypothesis, which has not been proven up to now [1, 2].

Different motion equations should refer to different groups of fluids with different names but there are currently no recommendations for correct terminology that takes into consideration differences in mathematical limitations.

Table 1

Limitations for deriving equations (1) and (2)

No.	Limitation for Navier equation	Motion equation	Name of fluid
1	$\tau=\mu\cdot\text{grad}u$; $p_{xx}=-p+2\mu\frac{\partial u_x}{\partial x}$; $p_{yy}=-p+2\mu\frac{\partial u_y}{\partial y}$; $p_{zz}=-p+2\mu\frac{\partial u_z}{\partial z}$	(1)	Stokes (Newtonian)
2	$\tau=\mu\cdot\text{grad}u$	(2)	Newtonian

Important for the analysis of currents is an equation for the full derivative from speed in the Gromeka-Lamb form. This notation is equivalent to a standard formula but makes it possible to determine the effect of linear and angular velocity on the full acceleration of the particle (*du/dt*).

In a vector form, this equation is recorded as follows:

$$\frac{du}{dt}=\frac{\partial u}{\partial t}+grad\left(\frac{u^2}{2}\right)+2\cdot[\vec{\omega}\times\vec{u}]. \tag{3}$$

In projections onto the coordinate axes:

$$\frac{du_x}{dt}=\frac{\partial u_x}{\partial t}+\frac{\partial}{\partial x}\left(\frac{u^2}{2}\right)+2\cdot\left(u_z\omega_y-u_y\omega_z\right),$$

$$\frac{du_y}{dt}=\frac{\partial u_y}{\partial t}+\frac{\partial}{\partial y}\left(\frac{u^2}{2}\right)+2\cdot\left(u_x\omega_z-u_z\omega_x\right),$$

$$\frac{du_z}{dt}=\frac{\partial u_z}{\partial t}+\frac{\partial}{\partial z}\left(\frac{u^2}{2}\right)+2\cdot\left(u_y\omega_x-u_x\omega_y\right).$$

The convective part of the full acceleration in (3) also follows from the vector analysis formula $(\vec{u}\cdot\nabla)\vec{u}=\text{grad}(u^2/2)+rot\,\vec{u}\times\vec{u}$.

Equation (3) must satisfy complete derivatives in all motion equations regardless of accounting (non-accounting) the viscosity but they are used only in the Euler equation for the ideal liquid.

Full acceleration in Navier equations (1) and (2) must also satisfy expression for a full derivative in form (3). It follows from the formal recording of equation (1) that the Laplace operator on speed depends only on one argument (u) while the full acceleration – on two arguments (u and ω). Thus, from the standpoint point of physics, $\nabla^2 u$ characterizes the laminar mode of the flow, and full acceleration –turbulent. The lack of a consistent effect of u and ω on both components makes it difficult to derive a common solution to the Stokes equation.

Paper [8] analyzes the issues related to the Stokes equation and gives examples of misconceptions (contradictions) in theoretical hydrodynamics. It is noted that one of the common problems is the description of vortex flows arising under the influence of viscosity and inertia forces; a model of streamlining a thin horizontal plate was also suggested. A given model produces a partial description of the flow and uses simplifications that do not make it possible to take into consideration the impact of all existing modes.

Issues with the solution to equation (1) led to the development of new models and equations that have a limited scope of application. One of these models (Birkhoff-Rott equation) is used to analyze the rotation of wind generator blades [9]. A characteristic feature of such a process is the rotation of flow particles, which is taken into consideration in an indirect way. The developed model yields a satisfactory result but is difficult for engineering use and is characterized by numerous limitations.

For complex processes, the motion equation is used within the system of equations in conjunction with the equations of heat exchange and electromagnetism. In work [10], a model of the numerical solution to the problem of convection in a vessel for storing cryogenic fluids has been developed. The effect of particle rotation is taken into consideration in an indirect way as this factor is absent from equation (1). Implementing this model requires the development of a separate computer program, making it difficult to use it.

Work [11] models a system of equations in which there is a special case of the Stokes equation without viscosity and equation of electromagnetism. This simplification provides an approximate pattern of the distribution of mass and charge streams only for the two-dimensional current model in the accelerator. The results of the calculation require additional refinement of the process parameters with the help of the experiment.

The shortcomings of the analyzed works are caused by the lack of a correct model of fluid movement and the complexity of solving the nonlinear equation (1). That has led to the emergence of simplified theoretical or semi-empirical models that allow only special problems to be solved in a narrow range of influencing factors.

The Stokes equation is close to the law of motion amount preservation, is common, and claims to be the primary in fluid mechanics [12]. The current paper addresses some of the contradictions in this equation, as well as techniques to remove them. That would clarify the limits of its applicability in terms of mathematics and physics.

3. The aim and objectives of the study

The aim of this study is to derive motion equations based on the consistent influence of linear and angular velocity on the components that take into consideration friction and inertia. This would make it possible to build more complete and accurate mathematical models, which could expand the range of problems to be solved and improve their quality.

To accomplish the aim, the following tasks have been set:

– to analyze a Laplace's operator on speed, establish its dependence on ω, and find a new form of the Stokes equation and its special cases;

– to solve and analyze two special problems.

4. Analysis of motion equations

4. 1. Laplace operator and Stokes equation. Special cases

Bring the Stokes equation to a form more convenient for later analysis and combine components that take into consideration viscosity. Perform the necessary transformations for the x coordinate.

– for the limitation of normal stress (Table 1):

$$p_{xx}=-p+2\mu\frac{\partial u_x}{\partial x}=-p_x \text{ or } p=p_x+2\mu\frac{\partial u_x}{\partial x},$$

where the normal stress $p_{xx}=-p_x$ according to the sign rule.

Then, the pressure component takes the following form:

$$-\frac{1}{\rho}\frac{\partial p}{\partial x}=-\frac{1}{\rho}\frac{\partial}{\partial x}\left(p_x+2\mu\frac{\partial u_x}{\partial x}\right)=-\frac{1}{\rho}\frac{\partial p_x}{\partial x}-2\nu\frac{\partial^2 u_x}{\partial x^2}; \tag{4}$$

– transform the Laplace operator and separate the components that take into consideration the influence of linear and angular velocity.

Then

$$\nabla^2 u_x=\frac{\partial^2 u_x}{\partial x^2}+\frac{\partial^2 u_x}{\partial y^2}+\frac{\partial^2 u_x}{\partial z^2}.$$

Express the second and third terms through the first derivative, add zero in brackets, and represent it as two identical terms with different signs.

$$\frac{\partial^2 u_x}{\partial y^2}=\frac{\partial}{\partial y}\left(\frac{\partial u_x}{\partial y}-\frac{\partial u_y}{\partial x}+\frac{\partial u_y}{\partial x}\right)=$$
$$=\frac{\partial^2 u_y}{\partial x\partial y}-\frac{\partial(rot\ u)_z}{\partial y}=\frac{\partial^2 u_y}{\partial x\partial y}-2\frac{\partial \omega_z}{\partial y},$$

$$\frac{\partial^2 u_x}{\partial z^2}=\frac{\partial}{\partial z}\left(\frac{\partial u_x}{\partial z}-\frac{\partial u_z}{\partial x}+\frac{\partial u_z}{\partial x}\right)=$$
$$=\frac{\partial^2 u_z}{\partial x\partial z}+\frac{\partial(rot\ u)_y}{\partial z}=\frac{\partial^2 u_z}{\partial x\partial z}+2\frac{\partial \omega_y}{\partial z}.$$

It follows from these equations that there is a function $\psi(u, \omega)$, which depends on two arguments and has a component on the x axis in the following form:

$$\psi_x(u,\omega)=\frac{\partial^2 u_x}{\partial x^2}+\frac{\partial^2 u_y}{\partial x\partial y}+\frac{\partial^2 u_z}{\partial x\partial z}+2\left(\frac{\partial \omega_y}{\partial z}-\frac{\partial \omega_z}{\partial y}\right).$$

Taking into consideration the last equation and (4), the following is obtained:

$$-\frac{1}{\rho}\frac{\partial p}{\partial x}+\nu\cdot\nabla^2 u_x=-\frac{1}{\rho}\frac{\partial p_x}{\partial x}+$$
$$+\nu\left[-\frac{\partial^2 u_x}{\partial x^2}+\frac{\partial^2 u_y}{\partial x\partial y}+\frac{\partial^2 u_z}{\partial x\partial z}+2\left(\frac{\partial \omega_y}{\partial z}-\frac{\partial \omega_z}{\partial y}\right)\right].$$

The expression in brackets is a function of two arguments – $\phi_x(u, \omega)$. Performing similar transforms for the y and z axes, the following is obtained:

$$\varphi_x(u,\omega)=-\frac{\partial^2 u_x}{\partial x^2}+\frac{\partial^2 u_y}{\partial x\partial y}+\frac{\partial^2 u_z}{\partial x\partial z}-2(rot\ \omega)_x,$$

$$\varphi_y(u,\omega)=\frac{\partial^2 u_x}{\partial x\partial y}-\frac{\partial^2 u_y}{\partial y^2}+\frac{\partial^2 u_z}{\partial y\partial z}-2(rot\ \omega)_y, \tag{5}$$

$$\varphi_z(u,\omega)=\frac{\partial^2 u_x}{\partial x\partial z}+\frac{\partial^2 u_y}{\partial y\partial z}-\frac{\partial^2 u_z}{\partial z^2}-2(rot\ \omega)_z,$$

where

$$(rot\ \omega)_x = \frac{\partial\omega_z}{\partial y} - \frac{\partial\omega_y}{\partial z};$$

$$(rot\ \omega)_y = \frac{\partial\omega_x}{\partial z} - \frac{\partial\omega_z}{\partial x};$$

$$(rot\ \omega)_z = \frac{\partial\omega_y}{\partial x} - \frac{\partial\omega_x}{\partial y}.$$

Given (5) and a complete derivative in form (3), the Stokes equation can be written:

$$X - \frac{1}{\rho}\frac{\partial p_x}{\partial x} + \nu\cdot\left[-\frac{\partial^2 u_x}{\partial x^2} + \frac{\partial^2 u_y}{\partial x\partial y} + \frac{\partial^2 u_z}{\partial x\partial z} - 2(rot\ \omega)_x\right] - -\frac{\partial}{\partial x}\left(\frac{u^2}{2}\right) = \frac{\partial u_x}{\partial t} + 2(u_z\omega_y - u_y\omega_z),$$

$$Y - \frac{1}{\rho}\frac{\partial p_x}{\partial y} + \nu\cdot\left[\frac{\partial^2 u_x}{\partial x\partial y} - \frac{\partial^2 u_y}{\partial y^2} + \frac{\partial^2 u_z}{\partial y\partial z} - 2(rot\ \omega)_y\right] - -\frac{\partial}{\partial y}\left(\frac{u^2}{2}\right) = \frac{\partial u_y}{\partial t} + 2(u_x\omega_z - u_z\omega_x), \tag{6}$$

$$Z - \frac{1}{\rho}\frac{\partial p_z}{\partial z} + \nu\cdot\left[\frac{\partial^2 u_x}{\partial x\partial z} + \frac{\partial^2 u_y}{\partial y\partial z} - \frac{\partial^2 u_z}{\partial z^2} - 2(rot\ \omega)_z\right] - -\frac{\partial}{\partial z}\left(\frac{u^2}{2}\right) = \frac{\partial u_z}{\partial t} + 2(u_y\omega_x - u_x\omega_y).$$

In this notation, the terms that take into consideration viscous friction and inertia have the same influencing factors $-(u,\omega)$. In a short form, system (6) can be written as follows:

$$G - \frac{1}{\rho}\mathrm{div}p + \nu\cdot\varphi(u,\omega) - \mathrm{grad}\left(\frac{u^2}{2}\right) = \frac{\partial u}{\partial t} + 2[\vec{\omega}\times\vec{u}]. \tag{7}$$

It follows from equation (7) that the Stokes equation describes a turbulent flow mode within the average model.

A given equation was derived without additional restrictions. This means that (7) is another form of the Stokes equation notation.

When the viscosity is excluded (ν=0), a general equation for a non-viscous current is obtained:

$$G - \frac{1}{\rho}\mathrm{div}p - \mathrm{grad}\left(\frac{u^2}{2}\right) = \frac{\partial u}{\partial t} + 2[\vec{\omega}\times\vec{u}]. \tag{8}$$

If the tangent stresses are excluded from the Navier equation, (8) is obtained as well.

Consider the special cases of Stokes equation in form (6):

1. For the laminar current mode, the angular velocity $\omega(x,y,z)=0$, so the equation takes the following form:

$$X - \frac{1}{\rho}\frac{\partial p_x}{\partial x} + \nu\cdot\left(-\frac{\partial^2 u_x}{\partial x^2} + \frac{\partial^2 u_y}{\partial x\partial y} + \frac{\partial^2 u_z}{\partial x\partial z}\right) - -\frac{\partial}{\partial x}\left(\frac{u^2}{2}\right) = \frac{\partial u_x}{\partial t},$$

$$Y - \frac{1}{\rho}\frac{\partial p_y}{\partial y} + \nu\cdot\left(\frac{\partial^2 u_x}{\partial x\partial y} - \frac{\partial^2 u_y}{\partial y^2} + \frac{\partial^2 u_z}{\partial y\partial z}\right) - -\frac{\partial}{\partial y}\left(\frac{u^2}{2}\right) = \frac{\partial u_y}{\partial t}, \tag{9}$$

$$Z - \frac{1}{\rho}\frac{\partial p_z}{\partial z} + \nu\cdot\left(\frac{\partial^2 u_x}{\partial x\partial z} + \frac{\partial^2 u_y}{\partial y\partial z} - \frac{\partial^2 u_z}{\partial z^2}\right) - -\frac{\partial}{\partial z}\left(\frac{u^2}{2}\right) = \frac{\partial u_z}{\partial t}.$$

In a short notation, system (9) takes the following form:

$$G - \frac{1}{\rho}\mathrm{div}p + \nu\cdot\varphi(u) - \mathrm{grad}\left(\frac{u^2}{2}\right) = \frac{\partial u}{\partial t}. \tag{10}$$

2. At $u(x,y,z)=0$, (6) produces a system of equations for a standing vortex

$$X - \frac{1}{\rho}\frac{\partial p_x}{\partial x} - 2\nu\cdot(rot\ \omega)_x = \frac{\partial(\omega_x r_{yz})}{\partial t},$$

$$Y - \frac{1}{\rho}\frac{\partial p_y}{\partial y} - 2\nu\cdot(rot\ \omega)_y = \frac{\partial(\omega_y r_{xz})}{\partial t},$$

$$Z - \frac{1}{\rho}\frac{\partial p_z}{\partial z} - 2\nu\cdot(rot\ \omega)_z = \frac{\partial(\omega_z r_{xy})}{\partial t}.$$

Or, in a short form,

$$G - \frac{1}{\rho}\mathrm{div}p - 2\nu\cdot(rot\ \omega) = \frac{\partial(\omega r)}{\partial t}. \tag{11}$$

At ν=0, (10) and (11) produce motion equations for a non-viscous current model:

$$G - \frac{1}{\rho}\mathrm{div}p - \mathrm{grad}\left(\frac{u^2}{2}\right) = \frac{\partial u}{\partial t}. \tag{12}$$

$$G - \frac{1}{\rho}\mathrm{div}p = \frac{\partial(\omega r)}{\partial t}. \tag{13}$$

Equations (12) and (13) characterize the linear current without inertial vortexes and a non-viscous standing vortex, respectively.

4. 2. Special problems

The following special problems have been selected: the established turbulent current on a horizontal plate and in a horizontal circular tube. The goal of solving both problems is to find the distribution of speed along the normal to the surface.

There are two ways to find solutions. The first technique employs the Stokes equation while the second involves the Navier equation. Both differential equations are simplified and integrated.

The first technique finds the distribution of speed by integrating the one-dimensional motion equation of the second order. The second technique finds the distribution of the tangent stress and then the distribution of speed using Newton's law for viscous friction. Both techniques complement each other and should produce the same result. It is assumed, in this case, that the liquid is incompressible and the thermal-physical properties are constant.

Consider the current on a horizontal plate at a turbulent boundary layer (Fig. 1).

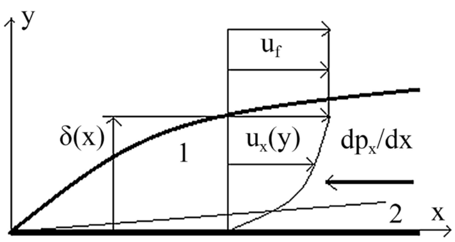

Fig. 1. Estimation scheme of the current on a plate: 1 – turbulent boundary layer; 2 – laminar underlay

The Stokes equation in form (1) shall be used, which, for a given case, takes the following form:

$$\frac{d^2u_x}{dy^2}=\frac{1}{\mu}\frac{dp}{dx}.$$

After double integration, the following is obtained:

$$u_x(y)=\frac{1}{2\mu}\frac{dp}{dx}y^2+c_1y+c_2. \tag{14}$$

The Navier equation shall be used to find the distribution of the tangent stress. For the x coordinate, ($p_{xx}=-p_x$) is obtained:

$$X-\frac{1}{\rho}\frac{\partial p_x}{\partial x}+\frac{1}{\rho}\left(\frac{\partial\tau_{yx}}{\partial y}+\frac{\partial\tau_{zx}}{\partial z}\right)=\frac{du_x}{dt}.$$

Following the simplification in accordance with earlier assumptions:

$$\frac{dp_x}{dx}=\frac{d\tau_{yx}}{dy}.$$

After the integration at dp_x/dx=const, find:

$$\tau_{yx}=\frac{dp_x}{dx}y+c_1.$$

The distribution of speed along the normal to the surface of the plate is determined from the following equation:

$$\mu\frac{du_x}{dy}=\frac{dp_x}{dx}y+c_1.$$

Following the integration $[(1/\mu)\cdot(dp_x/dx)=\text{const}]$, equation (14) is obtained.

Fig. 2 shows the scheme to find common integrals for the distribution of the tangent stress and speed for a flow on the plate.

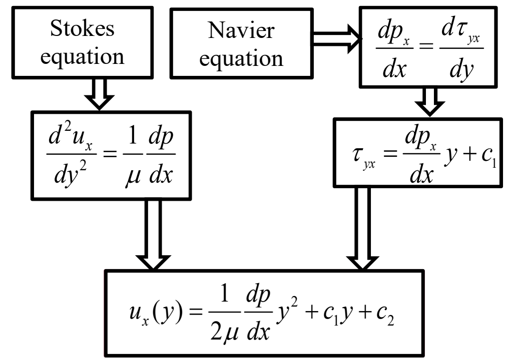

Fig. 2. Scheme to find integrals for a turbulent current on the plate

To find the distribution of speed in the laminar sublayer, one needs to use equation (9). However, it lacks the term d^2u_x/dy^2. This means that it is impossible to find a speed distribution for this part of the current.

Find a special solution to equation (14) for the following boundary conditions: at $y=\delta(x)$, $\tau_x(y)=0$, and $u_x(y=\delta)=u_f$.

The following is then obtained:

$$u_x(y)=\frac{1}{2\mu}\frac{dp_x}{dx}\left[y^2+\delta(x)^2-2y\cdot\delta(x)\right]+u_f. \tag{15}$$

Fig. 3 shows a comparison of the speed distribution from equation (15) and the known power distribution/$ux1(y)$/ $u_x(y)=u_f[y/\delta(x)]^{1/7}$ [1].

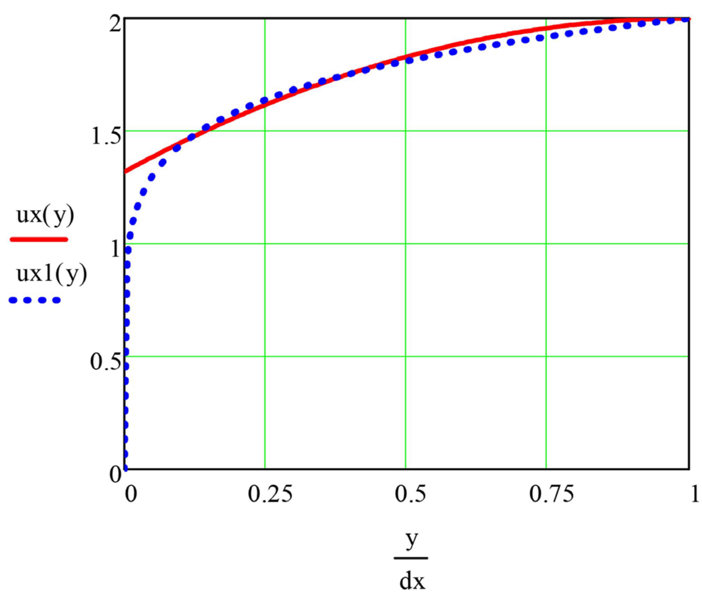

Fig. 3. Comparison of theoretical solution (15) with a semi-empirical equation: $(1/2\mu)dp_x/dx=-1{,}700$ $(m\cdot s)^{-1}$

It follows from Fig. 3 that the speed distribution (15) is consistent with the experiment only in the central part. That can be explained by a change in the current mode near the wall from turbulent to laminar, for which there is no analytical solution to the Stokes equation. Deviation from the empirical equation occurs when $y/\delta(x)<0.1$.

Consider the current in a straight circular pipe and find common integrals for the distribution of tangential stress and speed along the radius of the pipe (Fig. 4).

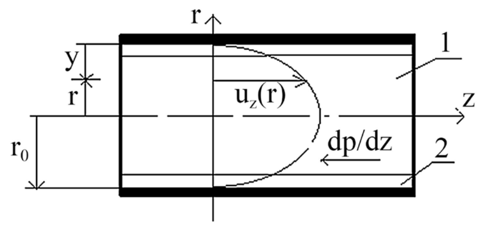

Fig. 4. Estimation scheme of the current in a pipe: 1 – turbulent core; 2 – laminar underlay

The Stokes equation in form (1) in the (r, z) coordinates is applied next. Since the current is steady and one-dimensional, the equation takes the following form:

$$\frac{d^2u_z}{dr^2}+\frac{1}{r}\frac{du_z}{dr}=\frac{1}{\mu}\frac{dp}{dz}.$$

The double integration $\left[(1/\mu)(dp/dz)=\mathrm{const}\right]$ produces the following:

$$u_z(r)=\frac{1}{4\mu}\frac{dp}{dz}r^2+c_1\ln r+c_2. \qquad (16)$$

Solve the same problem using the Navier equation.
From the equation in terms of stresses [1–3]:

$$Z-\frac{1}{\rho}\frac{\partial p_z}{\partial z}+\frac{1}{\rho}\left(\frac{\partial\tau_{zr}}{\partial r}+\frac{1}{r}\frac{\partial\tau_{\theta r}}{\partial\theta}+\frac{\tau_{zr}}{r}\right)=\frac{du_z}{dt}, \qquad (17)$$

where p_z is the pressure along the z axis, which, according to the rule of signs, is opposite to the normal stress p_{zz}.

Simplify equation (17) believing that there are no mass forces, no rotation of the flow around the pipe axis.

The following is then obtained:

$$\frac{\partial\tau_{zr}}{\partial r}+\frac{\tau_{zr}}{r}=\frac{\partial p_z}{\partial z}. \qquad (18)$$

With a constant diameter of the pipe (dp_z/dz=const), the solution to equation (18) takes the following form:

$$\tau_{zr}=\frac{c_1}{r}+\frac{{dp_z}/{dz}\cdot r}{2}.$$

Apply the Newton's equation $\tau_{zr}=\mu\cdot\frac{du_z}{dr}$:

$$\mu\frac{du_z}{dr}=\frac{1}{2}\frac{dp_z}{dz}r+\frac{c_1}{r}.$$

After the integration, equation (16) is obtained.

Fig. 5 shows the scheme to find integral (16) in two ways.

Find a special solution to equation (16) under the following boundary conditions: at $y=r_0$, $\tau=0$, and $u_z(y=r_0)=u_{\max}$.

The following is then obtained:

$$u_z(y)=\frac{1}{4\mu}\frac{dp_z}{dz}\left(y^2-r_0^2+2r_0^2\ln\frac{r_0}{y}\right)+u_{\max}. \qquad (19)$$

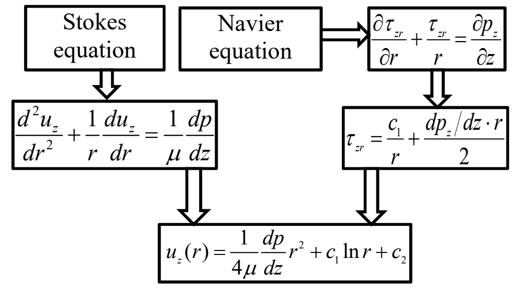

Fig. 5. Scheme to find common integrals for the turbulent current in a pipe

It follows from (19) that the speed on the wall $u_z(y=0)$ cannot be zero. This means that a given equation cannot be used for the near-wall laminar layer.

Fig. 6 shows a comparison of the speed distribution from equation (19) with the power law for the round pipe $u_z(y)=u_{\max}[y/r_0]^{0.16}$ [1].

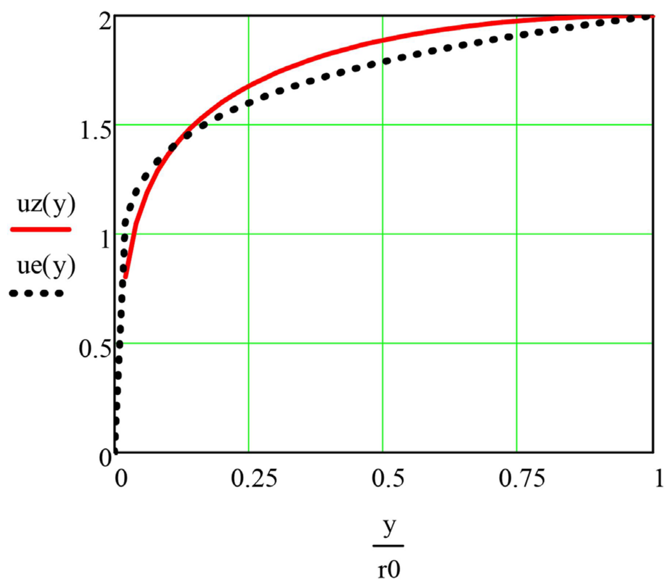

Fig. 6. Comparison of the theoretical distribution of speed in a pipe (red line) with a power semi-empirical equation (points): $u_{\max}$=2 m/s, $(1/4\mu)\cdot dp_z/dz=-70$ $(\mathrm{m\cdot s})^{-1}$

It follows from Fig. 6 that the theoretical distribution yields an inflated speed value (the order of about 10 %) at $y/r_0>0.12$. At $y/r_0<0.12$, the theoretical distribution tends to the finite speed on the wall, which does not correspond to the sticking hypothesis [1, 2].

Thus, the comparative analysis of two theoretical solutions with the semi-empirical equations reveals the same qualitative result despite the different integrals. In the central part of the flow, there is a satisfactory correspondence with the experiment, and near the wall – a significant deviation.

Theoretical curves fall into the region of empirical equations only at small constants' values, which are not typical of currents under normal conditions (the difference is 10–100 times). It is possible to achieve the estimated values of the constants $[(1/2\mu)\cdot(dp_x/dx)$=const and $(1/4\mu)\cdot(dp_z/dz)$==const$]$ with a combination of thermal-physical properties that are present in a rarefied gas (small density and relatively high viscosity). At low pressure, there is a molecular-viscosity mode (Knudsen number Kn ~0.1) under which the hypothesis of sticking does not hold, and the distribution of velocities near the wall is consistent with Fig. 3 and Fig. 6 [13, 14].

Thus, it is a relevant area of the experimental study to test the assumption that a rarefied gas is the Stokes liquid.

5. Discussion of the results of mathematical notation

When deriving Stokes equation, Newton's law for viscous friction in the form of $\tau=\mu\cdot\mathrm{grad}u$ is used. Such a notation does not demonstrate a sign of applicability to the turbulent mode of the current. To tackle this contradiction, Newton's equation should be transformed relative to the one-dimensional flow around a flat plate (Fig. 3).

Then

$$\tau=\mu\cdot\left(\frac{du_x}{dy}-\frac{du_y}{dx}+\frac{du_y}{dx}\right)=\mu\cdot\left(\frac{du_y}{dx}-2\omega_z\right), \qquad (20)$$

where

$$(rot\ u)_z=\frac{du_y}{dx}-\frac{du_x}{dy}=2\omega_z.$$

The presence in (20) of the linear and angular velocity indicates that Newton's friction law is true for two modes of flow – laminar and turbulent. The same conclusion follows from the analysis of the three-dimensional version of Newton's law [4].

The analysis reported here has revealed that there are three special cases of the Stokes equation, two of which were obtained as a result of the application of restrictions for the viscous liquid model ($\omega=0$ and $u=0$).

Using the condition $\nu=0$ produces a general equation for non-viscous liquid (8). The same equation follows from the Navier equation when excluding tangent stresses (τ_{ij}) and using a full derivative from speed in form (3).

Table 2 gives motion equations for viscous currents under different modes, as well as their analogs for the model of non-viscous liquid.

Fig. 7 shows the block diagram of decomposing the Navier and Stokes equations based on the conditions given in Table 2. There is no Euler equation for ideal liquid in this scheme as it requires additional assumptions for the hydrostatic pressure distribution law ($p_x=p_y=p_z=p$).

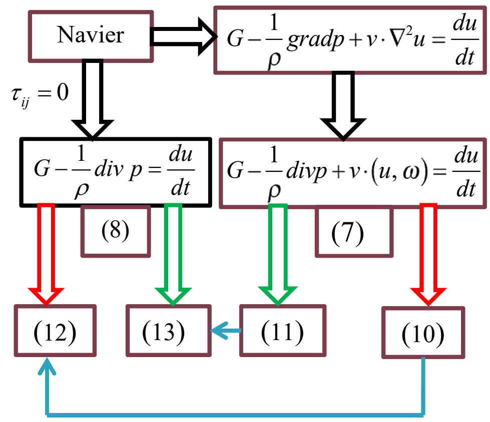

Fig. 7. Block diagram of decomposing the equations for viscous and non-viscous liquid. Conditions: red arrows – $\omega=0$; green – $u=0$; blue – $\nu=0$

The equations in Table 2 contain six unknowns (p_x, p_y, p_z, u_x, u_y, u_z) and are not closed. This property limits the possibilities of using the equations considered because it makes it possible to solve only one-dimensional problems. Thus, it is a relevant area of mathematical research to study ways to resolve the issue related to closing them.

6. Conclusions

1. Using the Gromeka-Lamb equation to find a complete acceleration du/dt, as well as the transformation of the Laplace operator, has made it possible to find the effect of the linear and angular velocity of particles on the Stokes equation. Applying the conditions for the non-vortex current ($\omega=0$), for the standing vortex ($u=0$), and for the model of non-viscous liquid ($\nu=0$) has made it possible to draw up a scheme of the Stokes equation decomposition and compare it with the special cases of the Navier equation.

Taking into consideration the influence of angular particle velocity makes it possible to more fully describe the flow of Newtonian (Stokes) fluid, as well as to find new methods of solving motion equations.

2. The comparison of the special solutions to the Stokes equation for a horizontal plate and a pipe with the semi-empirical equations has justified the assumption that a rarefied gas is the Stokes liquid. Experimental confirmation of this assumption may lead to practical applications in the field of vacuum technology.

Table 2

Motion equations for viscous and non-viscous currents

Equation	$u(x,y,z)\neq0$ $\omega(x,y,z)\neq0$	$u(x,y,z)\neq0$ $\omega(x,y,z)=0$	$u(x,y,z)=0$ $\omega(x,y,z)\neq0$
(1)	$G-\frac{1}{\rho}\mathrm{div}p+\nu\cdot\varphi(u,\omega)=\frac{du}{dt}$	$G-\frac{1}{\rho}\mathrm{div}p+\nu\cdot\varphi(u)-\mathrm{grad}\left(\frac{u^2}{2}\right)=\frac{\partial u}{\partial t}$	$G-\frac{1}{\rho}\mathrm{div}p+\nu\cdot\varphi(\omega)=\frac{\partial(\omega r)}{\partial t}$
Navier ($\tau_{ij}=0$)	$G-\frac{1}{\rho}\mathrm{div}p=\frac{du}{dt}$	$G-\frac{1}{\rho}\mathrm{div}p-\mathrm{grad}\left(\frac{u^2}{2}\right)=\frac{\partial u}{\partial t}$	$G-\frac{1}{\rho}\mathrm{div}p=\frac{\partial(\omega r)}{\partial t}$

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