Modelling Systems with Elements in Several States

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Abstract: - In this paper, we consider systems with one resource, which can be in several states. The states differ significantly in their processes of mortality, reproduction and mutual influence. For instance, infected elements can have a higher mortality rate than healthy and recovered ones. For cyclic models, in which the initial state of the system coincides with the final state, balance relations are found. They represent a system with functional operators with shift and integrals with degenerate kernels. Modified Fredholm method, proposed in previous works to solve the integral equations of the second type with degenerate kernels and shifts, is applied. Equilibrium position of a system with a three-state resource is found.

Key-Words: - Multiple states, Functional operator with shift, Inverse operator, Integral with degenerate kernel, Equilibrium position.

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1 Introduction

In works [1-6], we have studied systems with renewable resources. In modelling, we followed principles the main idea of which was to separate the individual and the group parameters and to provide a discretization of time. These approaches led us to the need for the mathematical apparatus of functional operators with shift. Some models with renewable resources were elaborated.

Here, we consider a system with one type of elements, which, however, can be in three states. It is possible to interpret the states as healthy elements being in the first state; sick, affected elements being in the second state and recovered elements with immunity being in the third state. Elements of the system that are in different state have different characteristics. Their mortality coefficients, terms of reproduction and processes of change of individual parameters (such as weight gain and loss) are not the same.

This work extends research carried out in [1], [3]. We follow the modelling principles and the approaches suggested in these works. Briefly, they consist in tracking the system at fixed time points and in studying dependence in the densities of group parameters distributed by individual parameters. For example, we are interested not only in the total volume of caught fish, but also in the distribution of the volumes of caught fish by weight; not only in the total number of infected patients, but also in the distribution of their numbers by weight or age. Some relevant studies can be found in [7] and [8]. Recently, scientific research on the mathematical modelling and control of communicable diseases has revived. We point out, for example [9, 10, 11]. Our theme is related to these studies.

In Section 2, we analyze results of system evolution and give a description and an interpretation of the coefficients and terms in the balance equations.

In Section 3, state changes and transitions from one state to another are described, balance relations are derived and cyclical models are proposed.

In Section 4, using inverse operators in Hölder spaces, in which the models are considered, we reduce balance equations to integral equations of the second kind with degenerate kernels.

Fredholm method of solving such equations is applied and the equilibrium state of the system with three-state objects is found.

The theory of linear functional operators with shift is the appropriate mathematical instrument for the investigation of renewable systems with multi-state elements.

2 Evolution of a System during a Time Period

Following our approach proposed in [1], [3], we will not track how a system changes during the time period (t_0, t_0+T) at every moment, but will fix results only at the final time point $t=t_0+T$.

The initial state of the system S at time t_0 is represented by density functions of distribution of the group parameter by the individual parameter. For the first state, by the function $n(x,t_0)=n(x)$, $0<x<x_{\max}$; for the second state, by the function $k(x,t_0)=k(x)$, $0<x<x_{\max}$ and for the third state, by the function $m(x,t_0)=m(x)$, $0<x<x_{\max}$.

Modification in the distribution of the group parameter by the individual parameter is represented by a displacement. The state of the system S at the time $t=t_0+T$ is:

$$n(x,t_0+T)=\frac{d}{dx}\alpha(x)\cdot n(\alpha(x)),$$
$$k(x,t_0+T)=\frac{d}{dx}\beta(x)\cdot k(\beta(x)),$$
$$m(x,t_0+T)=\frac{d}{dx}\beta(x)\cdot m(\beta(x)).$$

We consider natural mortality with the coefficient $d_n(x)$ for the elements, which are in the first state (healthy organisms), and the mortality coefficient $d_k(x)$ for the elements which are in the second state (infected organisms).

We obtain the final state of the system at the moment t_0+T, which is described as follows:

$$n(x,t_0+T)=d_n(x)\frac{d}{dx}\alpha(x)\cdot n(\alpha(x))+\zeta_n(x),$$
$$k(x,t_0+T)=d_k(x)\frac{d}{dx}\beta(x)\cdot k(\beta(x))+\zeta_k(x),$$
$$m(x,t_0+T)=d_n(x)\frac{d}{dx}\alpha(x)\cdot m(\alpha(x))+\zeta_m(x).$$

Here, terms $\zeta_n(x)$, $\zeta_k(x)$, $\zeta_m(x)$ that do not depend on $n(x)$ and $k(x)$, but have an impact on them, have been introduced. They can reflect various changes occurring in the system S; for example, migration processes. We note that the presence of these terms does not affect the method and the technique applied in this work; however, there are models in which they play an essential role. Thus, in [4] some optimization problems were formulated in which these functions are involved in the cost mechanisms and in the control of systems with renewable resources. In this work, we leave these problems aside.

The process of reproduction will be represented by the terms:

$$N^R(x)=N^r(x)C_n^r,\quad C_n^r=\int_0^1 \rho_n^r(\tau)n(\tau)d\tau,$$
$$K^R(x)=K^r(x)C_k^r,\quad C_k^r=\int_0^1 \rho_k^r(\tau)k(\tau)d\tau,$$
$$M^R(x)=M^r(x)C_m^r,\quad C_m^r=\int_0^1 \rho_m^r(\tau)m(\tau)d\tau.$$

and the reciprocal influence of the states of elements will be represented by the terms

$$N^K(x)=N^k(x)C_n^k,\quad C_n^k=\int_0^1 \rho_n^k(\tau)k(\tau)d\tau,$$
$$K^N(x)=K^n(x)C_k^n,\quad C_k^n=\int_0^1 \rho_k^n(\tau)n(\tau)d\tau,$$
$$K^M(x)=K^m(x)C_k^m,\quad C_k^m=\int_0^1 \rho_k^m(\tau)m(\tau)d\tau,$$

$$M^K(x)=M^k(x)C_m^k,\quad C_m^k=\int_0^1 \rho_m^k(\tau)k(\tau)d\tau .$$

The absence of terms $N_M(x)$ and $M^N(x)$ is explained by the fact that at this stage of description of the system, we do not distinguish between the first state and the third state. We will take into account the presence of immunity in recovered individuals when we will describe changes occurring to objects in the second state. Here, we did not strive to describe the general form of the degeneracy of kernels but instead focused on keeping the essence of the approach and the clarity of presentation. In describing the evolution of the system S over the time interval $j=(t_0,t_0+T)$, we used works [2, 3] on modelling systems with two renewable resources. The state of the system is described as follows:

$$n(x,t_0+T)=d_n(x)\alpha'(x)n(\alpha(x))+ N^R(x)+N^K(x)+\zeta_n(x), \quad (1)$$

$$k(x,t_0+T)= d_k(x)\beta'(x)k(\beta(x))+K^R(x)+K^N(x)+K^M(x)+\zeta_k(x),$$

$$m(x,t_0+T)=d_m(x)\gamma'(x)m(\gamma(x))+ M^R(x)+M^K(x)+\zeta_m(x). \quad (2)$$

Let us introduce a coefficient $[1-\Theta(x)]$, which separates the proportion of patients who are in a hospital and receive full medical care:

$$[1-\Theta(x)]k(x,t_0+\mathrm{T})=[1-\Theta(x)]d_k(x)\ \beta'(x)\ k[\beta(x)].$$

However, there is a proportion of infected

$$[\Theta(x)]k(x,t_0+T)=[\Theta(x)]\tilde{d}_k(x)\tilde{\beta}'(x)k[\tilde{\beta}(x)]$$

who do not go to a doctor, who do not want or cannot receive inpatient treatment because of material or other reasons. For example, in extreme situations when hospitals are overcrowded and the sick cannot receive full service. They have another mortality rate and a different function of change in the individual parameter $\beta(x)\neq\tilde{\beta}(x)$. We plan to study this case in subsequent works.

Now, we consider an alternative description of this proportion. To describe this term, we use an integral like this:

$$K^K(x)=K^k(x)C_k^k,\quad C_k^k=\int_0^1 \rho_k^k(\tau)k(\tau)d\tau .$$

The term $K^K(x)$ is similar in structure to $N^K(x)$ but has a different meaning: $N^K(x)$ describes the influence of the second state on the first state and the function $K^K(x)$ itself depends on the distribution density $k(x)$ and affects further changes in $k(x)$: thus, the term $K^K(x)$ reflects the processes of self-influence. Here, the term $K^K(x)$ shows the dependence of the dynamics of change in the number of patients on the already existing total number of patients: when there are many patients, not all receive the same amount of help.

We finally write down the balance ratios at the very end of the period:

$$k(x,t_0+T)=[1-\Theta(x)]d_k(x)\beta'(x)k(\beta(x))+ K^R(x)+K^N(x)+K^K(x)+\zeta_k(x). \quad (3)$$

Equations (1), (2), (3) are called equilibrium proportions or balance equations.

To these relations, which describe the evolution of the system S, we add

$$\int_0^1 \rho(\tau)n(\tau)d\tau=C, \quad (4)$$

which represents the initial conditions.

3 Transitions of States, Balance Relationships and the Cyclic Model

Until now, we have not described how the transition from one state to another occurs. In this section, we will go over this process. Let us make some assumptions.

First, according to our approach, we observe the system (we take measurements, get information or carry out monitoring) not at every moment of time, but only at fixed time points. Thus, we assume that the change in the states of elements occurs when periods change, that is, also at time points $t=t_0+T$. The elements do not change their states during the interval $[t_0,t_0+T)$ and the transition from one state to another occurs at the time point $t = t_0 + T$.

Second, immunity protects against infection only during one period $\left[t_0, t_0 + T\right)$.

Third, the duration of the illness is shorter than the length of the period and, moreover, if a certain organism was initially sick or infected, then by the end of the period it either recovers or leaves the system. This is taken into account by the mortality rate and, therefore, the term indicates the number of recovered patients and the number of healthy individuals. Patients do not pass to the next period.

Fourth, newly born elements are considered healthy, and the term describing the mutual influence of elements in the first and in the second states is refered to the elements in the first state, to the density distribution of $n\left(x, t_0 + T\right)$.

Thus, the elements that were in the second state (sick) at the beginning of the period, by the end of this period either leave the system or recover and enter the third state (with immunity). The elements that were in the third state at the beginning of the period, by the end of this period either leave the system or enter the first state (healthy without immunity). The elements that were in the first state at the beginning of the period, by the end of this period either leave the system, preserve first state or enter the second state.

We represent the term $d_n\left(x\right)\alpha'\left(x\right)n\left[\alpha\left(x\right)\right]$ as a sum of two parts:

$$\left[\Lambda\left(x\right)\right]d_n\left(x\right)\alpha'\left(x\right)n\left[\alpha\left(x\right)\right]$$

is the proportion of organisms that were healthy at the beginning $t = t_0$, but became infected during the time period $\left[t_0,\ t_0 + T\right)$ and move to the second state (sick) at the beginning of the second period $t = t_0 + T$ and

$$\left[1 - \Lambda\left(x\right)\right]d_n\left(x\right)\alpha'\left(x\right)n\left[\alpha\left(x\right)\right]$$

is the proportion of organisms that were healthy at the beginning $t = t_0$ without immunity and preserve their state during the time period $\left[t_0,\ t_0 + T\right]$.

The term $d_m(x)\gamma'\left(x\right)m\left[\gamma\left(x\right)\right]$ is the proportion of organisms that were healthy at the beginning $t = t_0$ with immunity and preserve its state during the time period $\left[t_0,\ t_0 + T\right)$, but these elements have lost their immunity and move to the first state (healthy without immunity) at the beginning of the second period $t = t_0 + T$.

These processes are reflected in changes in balance equations. We obtain:

$$n\left(x, t_0 + T\right) = \left[1 - \Lambda\left(x\right)\right]d_n\left(x\right)\alpha'\left(x\right)n\left[\alpha\left(x\right)\right] + \\ + d_m(x)\gamma'\left(x\right)m\left(\gamma\left(x\right)\right) + \\ N^R\left(x\right) + N^K\left(x\right) + K^R\left(x\right) + K^N\left(x\right) + M^R\left(x\right) + \zeta(x), \quad (5)$$

where $\zeta\left(x\right) = \zeta_n\left(x\right) + \zeta_m\left(x\right)$ and

$$k\left(x, t_0 + T\right) = \left[\Lambda\left(x\right)\right]d_n\left(x\right)\alpha'\left(x\right)n\left[\alpha\left(x\right)\right], \quad (6)$$

$$m\left(x, t_0 + T\right) = \\ [1 - \Theta\left(x\right)]d_k(x)\beta'\left(x\right)k\left(\beta\left(x\right)\right) + K^K\left(x\right) + \zeta_k. \quad (7)$$

Let us simplify our model. We assume that the affected and the recovered individuals do not reproduce and that their states do not affect each other:

$$K^R\left(x\right) = K^N\left(x\right) = N^K\left(x\right) = M^K\left(x\right) = M^R\left(x\right) = 0.$$

Without loss of generality, we suppose below that $x_{\max} = 1, \quad y_{\max} = 1$.

These constraints, as well as the simplified choice of degeneracy kernels, do not affect the methods used, but allow us to avoid unnecessary encumberment in the content of the article.

We note that different states of elements in this work are similar to different resources in our previous works. An essential difference is that representatives of the same resource cannot transform into representatives of another resource and thus do not have different states.

Let our goal be to find the equilibrium state of the system S, that is, to find such an initial distribution of group parameters by the individual parameters $n\left(x\right),\ k\left(x\right)$, that after all transformations during the time interval $\left(t_0, t_0 + T\right)$, it would coincide with the final distribution:

$$n\left(x, t_0\right) = n\left(x,\ t_0 + T\right),\ \ k\left(x, t_0\right) = k\left(x,\ t_0 + T\right).$$

From here and balance equations (5), (6), (7), it follows that

$$n(x)=\left[1-\Lambda(x)\right]d_n(x)\alpha'(x)n\left[\alpha(x)\right]+ d_m(x)\gamma'(x)m\left(\gamma(x)\right)+N^R(x)+N^K(x)+\zeta(x), \quad (8)$$

$$k(x)=\left[\Lambda(x)\right]d_n(x)\alpha'(x)n\left[\alpha(x)\right], \quad (9)$$

$$m(x)=[1-\Theta(x)]d_k(x)\beta'(x)k\left(\beta(x)\right)+K^K+\zeta_k. \quad (10)$$

This model is a cyclic model. Relations (8), (9), (10) and (4) are equilibrium proportions or balance equations of the cyclic model. The system of balance equations of the cyclic model is represented through lineal functional equations with shift and integrals with degenerate kernels. Theoretical mathematical results on the invertibility of functional operators with shift in Hölder space with shift [12] were just obtained by the authors of this article. And we, here, when modelling the systems under consideration, will use them. These results [12] will be the main mathematical tool for analyzing balance equations of our model.

The discretization of time and the special attention which we have paid to the study of density distributions of the group parameters by the individual parameters, lead us to functional operators with shift.

Consider two special cases: the case when the conditions

$$\beta(x)=x, \quad \gamma(x)=\alpha(x) \quad (11)$$

are fulfilled and the case when the conditions

$$\beta(x)=x, \quad \gamma(x)=x \quad (12)$$

are fulfilled. The equality $\beta(x)=x$ means that diseased individuals do not change their individual parameter. In this situation, survival is more important than change in weight. The equality $\gamma(x)=\alpha(x)$ means that healthy organisms, including recovered ones, equally change their individual parameter (weight) over time. The equality $\gamma(x)=x$ means that for the recovered organisms, despite the fact that they are already healthy, they need time for full rehabilitation, and during this period, their individual parameter does not change. Conditions (11) and (12) are fundamental and essential since the choice of the applied mathematical apparatus and research methods depends on it.

If requirements (11) are met, then relations (10) take the form:

$$m(x)=[1-\Theta]d_k\cdot k(x)+K^K(x)+\zeta_k.$$

From (8), we have

$$n(x)=\left[1-\Lambda\right]d_n\alpha'(x)n\left[\alpha(x)\right]+ d_m\alpha'(x)m\left(\alpha(x)\right)+N^R(x)+N^K(x)+\zeta(x),$$

and

$$n(x)=\left[1-\Lambda\right]d_n\alpha'(x)n\left[\alpha(x)\right]+d_m\alpha'(x)\times \left\{[1-\Theta(\alpha)]d_k(\alpha)\cdot k\left(\alpha(x)\right)+K^K\left(\alpha(x)\right)+\zeta_k(\alpha)\right\}+ N^R(x)+N^K(x)+\zeta(x).$$

From (9), we have

$$n(x)=\delta(x)n\left(\alpha(x)\right)+\eta(x)n\left(\alpha\left(\alpha(x)\right)\right)+ d_m\alpha'K_K\left(\alpha(x)\right)+d_m\alpha'\zeta\left(\alpha(x)\right)+ N^R(x)+N^K(x)+\zeta,$$

where $\delta(x)$ and $\eta(x)$ can be written out in terms of known functions from (8), (9), (10). To shorten the notation, the variables of some functions have been omitted. In this article, we will focus on one case (12), we plan to study case (11) in subsequent works.

If requirements (12) are met, then

$$n(x)=\left[1-\Lambda\right]d_n\alpha'(x)n\left[\alpha(x)\right]+d_m m(x)+ N^R(x)+N^K(x)+\zeta$$

and

$$n(x)-d_k\alpha'(x)\left\{\left[1-\Lambda\right]+d_m\left[1-\Theta\right]\Lambda d_n\right\}n\left[\alpha(x)\right]= d_mK^K(x)+d_m\zeta_k(x)+N^R(x)+N^K(x)+\zeta(x). \quad (13)$$

Rewriting equation (13) in operator form, we get

$$An(x)=d_mK^K+d_m\zeta_k+N^R(x)+N^K(x)+\zeta,$$

where

$$\left(An\right)(x)=n(x)-b(x)n[\alpha(x)]$$

and

$$b(x) = \left[1-\Lambda(x)\right] d_n(x) \alpha'(x) + [1-\Theta(x)] d_k(x) \Lambda(x) d_n(x) \alpha'(x).$$

We also express balance equations through the shift operator $(\mathrm{B}_\alpha \nu)(x) = \nu[\alpha(x)]$:

$$[I - b(x)\mathrm{B}_\alpha] n(x) = N^R(x) + N^K(x) + d_m(x) K^K(x) + h(x), \qquad (14)$$

$$k(x) = c(x)(\mathrm{B}_\alpha n)(x), \qquad (15)$$

where

$$c(x) = \Lambda(x) d_n(x) \alpha'(x),$$
$$h(x) = d_m(x)\, \zeta_k(x) + \zeta(x).$$

We will consider the model and its balance equations (14), (15) in the classes of Hölder functions and in spaces of Hölder functions with weights.

4 Reduction of Balance Relations to Fredholm Equations of Second Kind and Finding the Equilibrium State

We require that the shifts of individual parameters have properties that make it possible to apply the results on the invertibility of functional operators with shift.[12].

Let $\alpha(x)$ be a bijective orientation-preserving displacement on J: if $x_1 < x_2$ then $\alpha(x_1) < \alpha(x_2)$ and has only two fixed points $x = 0, x = 1$: $\alpha(0) = 0$, $\alpha(1) = 1$; $\alpha(x) \neq x$ when $x \neq 0$, $x \neq 1$. In addition, let $\alpha(x)$ be a differentiable function and $\frac{d}{dx}\alpha(x) \neq 0,\ x \in J,\ J = [0,1]$.

We recall the definitions of the space of Hölder class functions with weight [1], [3]: $H_\mu^0(J, \rho)$,

$$\rho(x) = x^{\mu_0}(1-x)^{\mu_1},\ 0 < \mu < 1,\ \mu < \mu_i < 1 + \mu,\ i = 0,1.$$

Functions $\varphi(x)$ that satisfy the following condition:

$$\varphi(x_1) - \varphi(x_2) | \leq C | x_1 - x_2 |^\mu, \quad x_1 \in J, x_2 \in J$$

are called Hölder functions and form the class $H_\mu(J)$. Functions that become Hölder functions and turn into zero at points $x = 0,\ x = 1$, after being multiplied by $\rho(x)$, form a Banach space. Functions of this space are called Hölder functions with weight $\rho(x)$ and have the notion $H_\mu^0(J, \rho)$. The norm in the space $H_\mu^0(J, \rho)$ is defined by

$$\| f(x) \|_{H_\mu^0(J,\rho)} = \| \rho(x)\, f(x) \|_{H_\mu(J)}$$

where

$$\| \rho(x)\, f(x) \|_{H_\mu(J)} = \| \rho(x)\, f(x) \|_C + \| \rho(x)\, f(x) \|_\mu,$$

$$\| \rho(x) f(x) \|_C = \max_{x \in J} |\rho(x) f(x)|,$$

$$\| \rho(x) f(x) \|_\mu = \sup_{x_{1,2} \in J,\, x_1 \neq x_2} \frac{|\rho(x_1) f(x_1) - \rho(x_2) f(x_2)|}{| x_1 - x_2 |^\mu}.$$

Let us formulate the invertibility conditions for functional equations with shift in the space $H_\mu^0(J, \rho)$ from [12]. We consider the operator:

$$(\tilde{A}\nu)(x) = \tilde{a}(x)(I\nu)(x) - \tilde{b}(x)(B_\alpha \nu)(x),$$

where I is the identity operator and B_α is the shift operator,

$$(B_\alpha \nu)(x) = \nu[\alpha(x)].$$

Functions $\tilde{a}(x)$ and $\tilde{b}(x)$ belong to Hölder class functions $H_\mu(J)$ with exponent $0 < \mu < 1$.

Operator $\tilde{A}$, acting in the Banach space $H_\mu^0(J, \rho)$ is invertible if the following condition is fulfilled:

$$\Phi_\alpha[\tilde{a}(x), \tilde{b}(x), H_\mu^0(J, \rho)] \neq 0, \quad x \in J,$$

where function Φ_α is defined by

$$\Phi_\alpha[\tilde{a}(x),\tilde{b}(x),H_\mu^0(J,\rho)]=$$

$$\begin{cases} \tilde{a}(x), & \text{when} \quad |\tilde{a}(0)|>|\tilde{a}'(0)|^{-\mu_0+\mu}|\tilde{b}(0)| \\ & \text{and when} \quad |\tilde{a}(1)|>|\tilde{a}'(1)|^{-\mu_0+\mu}|\tilde{b}(1)| \\ \tilde{b}(x), & \text{when} \quad |\tilde{a}(0)|<|\tilde{a}'(0)|^{-\mu_0+\mu}|\tilde{b}(0)| \\ & \text{and when} \quad |\tilde{a}(1)|<|\tilde{a}'(1)|^{-\mu_0+\mu}|\tilde{b}(1)| \\ 0 & \text{in all other cases.} \end{cases}$$

We return to balance relations (8), (9), (10). The functions, of which the known coefficients $b(x)$ from $A=I-b(x)$ and $c(x)$ from (11) are composed, are assumed to belong to $H_\mu(J)$ and functions $N^K(x), N^R(x), K^K(x), h(x), \zeta_n(x), \zeta_k(x), \zeta_m(x)$ belong to $H_\mu^0(J,\rho)$. We seek the required functions in the space $H_\mu^0(J,\rho)$.

If the distribution $n(x)$ can be found, it will not be difficult to calculate the distributions $k(x)$ and $m(x)$ so we will focus on analyzing equation (8).

For solving this system of equations, let us make use of the approach for the solution of integral Fredholm equations of the second type with degenerate kernels [13].

Let us assume that the invertibility conditions for the operator A in space $H_\mu^0(J,\rho)$ are satisfied:

$$\Phi_\alpha[1,b(x),H_\mu^0(J,\rho)]\neq 0, \quad x\in[0,1].$$

It follows that the operator A^{-1} inverse to the operator A exists. Different forms of inverse operator A^{-1} can be found in [3], [12]. Note that the series representing inverse operators in Lebesgue spaces with weight [14] are also suited for Hölder spaces with weight, when they converge.

First, let us apply operator A^{-1} to the left side of equations (9). We obtain balance equations in the form

$$n(x)=A^{-1}\left[N^R(x)+N^K(x)+d_m(x)K^K(x)+h(x)\right]$$

or

$$n(x)=C_n^r\left(A^{-1}N^r\right)(x)+C_n^k\left(A^{-1}N^k\right)(x)+ C_k^k A^{-1}\left(d_m(x)K^K\right)(x)+A^{-1}h(x), \tag{16}$$

$$k(x)=c(x)(\mathrm{B}_\alpha n)(x), \tag{17}$$

$$m(x)=[1-\Theta(x)]d_k(x)k(x)+K^K(x)+\zeta_k(x), \tag{18}$$

$$\int_0^1 \rho(\tau)n(\tau)d\tau=C. \tag{19}$$

Having multiplied the equation of system (16) by $\rho_n^r(x),\ \rho_n^k(x),\ \rho(x)$, and then having integrated over the interval J, we obtain:

$$\int_J \rho_n^r(x)n(x)\,dx=C_n^r\int_J \rho_n^r(x)\left(A^{-1}N^r\right)(x)dx+ C_n^k\int_J \rho_n^r\left(A^{-1}N^k\right)(x)\,dx+C_k^k\int_J \rho_n^r\left(A^{-1}K^k\right)(x)\,dx+g_1,$$

$$\int_J \rho_n^k(x)n(x)\,dx=C_n^r\int_J \rho_n^k(x)\left(A^{-1}N^r\right)(x)dx+ C_n^k\int_J \rho_n^k\left(A^{-1}N^r\right)(x)\,dx+C_k^k\int_J \rho_n^k\left(A^{-1}K^k\right)(x)\,dx+g_2,$$

$$\int_J \rho(x)n(x)\,dx=C_n^r\int_J \rho(x)\left(A^{-1}N^r\right)(x)dx+ C_n^k\int_J \rho(x)\left(A^{-1}N^r\right)(x)\,dx+C_k^k\int_J \rho(x)\left(A^{-1}K^k\right)(x)\,dx+g_3,$$

where

$$g_1(x)=\int_J \rho_n^r(x)\left(A^{-1}h\right)(x)dx,$$
$$g_2(x)=\int_J \rho_n^k(x)\left(A^{-1}h\right)(x)dx,$$
$$g_3(x)=\int_J \rho(x)\left(A^{-1}h\right)(x)dx-C.$$

Taking into account

$$\int_J \rho_n^r(x)n(x)dx=C_n^r, \quad \int_J \rho_n^k(x)n(x)dx=C_n^k$$

and the initial conditions $\int_J \rho(x)n(x)dx=C$, we represent (17) in the matrix form:

$$M\times X=G,$$

$$M = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{13} & a_{23} & a_{33} \end{bmatrix}, \quad X = \begin{bmatrix} C_n^r \\ C_n^k \\ C_k^k \end{bmatrix}, \quad G = \begin{bmatrix} g_1 \\ g_2 \\ g_3 \end{bmatrix},$$

where

$$\mathrm{a}_{11} = 1 - \int_J \rho_n^r \left(A^{-1}N^r\right)dx, \ \mathrm{a}_{11} = -\int_J \rho_n^r \left(A^{-1}N^k\right)dx,$$

$$\mathrm{a}_{13} = -\int_J \rho_n^r \left(A^{-1}d_m K^r\right)dx, \ \mathrm{a}_{21} = -\int_J \rho_n^k \left(A^{-1}N^r\right)dx,$$

$$\mathrm{a}_{22} = 1 - \int_J \rho_n^k \left(A^{-1}N^r\right)dx, \ \mathrm{a}_{23} = -\int_J \rho_n^k \left(A^{-1}d_m K^k\right)dx,$$

$$\mathrm{a}_{31} = -\int_J \rho\left(A^{-1}N^r\right)dx, \quad \mathrm{a}_{32} = -\int_J \rho\left(A^{-1}N^k\right)dx,$$

$$\mathrm{a}_{33} = -\int_J \rho\left(A^{-1}d_m K^k\right)dx.$$

Let us assume that the determinant of this system is different from zero, $det\, M \neq 0$. It follows that matrix M^{-1} inverse to the matrix M exists and can be constructed. Then, we can determine the unknown constants C_n^r, C_n^k, C_k^k and calculate the solution of balance system from (16), (17), (18) and (19). The unknown distributions of densities are:

$$n(x) = C_n^r \left(A^{-1}N^r\right)(x) + C_n^k \left(A^{-1}N^k\right)(x) + C_k^k A^{-1}\left(d_m(x)K^k\right)(x) + A^{-1}h(x),$$

$$k(x) = c(x)\mathrm{B}_\alpha[C_n^r \left(A^{-1}N^r\right)(x) + C_n^k \left(A^{-1}N^k\right)(x) + C_k^k A^{-1}\left(d_m(x)K^k\right)(x) + A^{-1}h(x)],$$

$$m(x) = [1-\Theta(x)]d_k(x)c(x)\mathrm{B}_\alpha[C_n^r \left(A^{-1}N^r\right)(x) + C_n^k\left(A^{-1}N^k\right)(x) + C_k^k A^{-1}\left(d_m(x)K^k\right)(x) + A^{-1}h(x)] + C_k^k K^k(x) + \zeta_k(x).$$

Thus, we have found the equilibrium state of the cyclic model of system S and obtained the state to which the system returns after the time interval T. Analysis of the distributions $n(x), k(x), m(x)$ will show us the state of our system at periodic check points and will help us determine whether it is necessary to improve the situation or whether it is within the limits of the norm.

5 Conclusion

In previous works, based on the theory of linear Operators with shift, we elaborated models for systems with renewable resources.

Applying the developed approach, in this work, we present cyclic models for systems with elements, which can be in different states: for example, infected or not infected or recovered with immunity.

In the following works, we plan to consider open non-cyclic models, which will allow us to determine the limit situation to which the system will tend.

We intend to move to more general relationships between shifts of the individual parameters than those covered in this article (11), (12), and this will certainly require further development of mathematical methods for the study of such systems.

Another direction for the further development of our approach can be further development of the ideas outlined in work [4]: based on the proposed model, formulate economic and environmental problems for the preservation and use of systems with renewble resources.

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