Use of Discrete and Continuous Markov Chains for System Absorbing States

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I. INTRODUCTION

Analysis of world experience has shown the feasibility of using several parameters to evaluate the effectiveness of projects, which allows solving the important tasks of meeting the requirements of project effectiveness in the conditions of limited time, financial, human and other types of resources most effectively [1–3].

The project approach, as a basis for change management, directs any activity to proactive (with prejudice) principles of system management "project – project team – environment" through the use of models that reflect the essential properties of the system, including methods of measuring project parameters and evaluation of their effectiveness [4 – 6].

In the case of solving the problem of estimating the production system in terms of value creation, the set of probabilities of certain states, which reflect the level of perfection of the system in the sense of compliance with some criteria, is selected as an objective function [7–8]. The system can be modified and improved by management. This is possible when using effects on resources, technologies, communications or structural changes in the system [9 – 10].

II. THE AIM OF THE ARTICLE

The article continues the research cited in [10 – 14]. In these works the use of different Markov chains for modeling of management processes of project-managed or project-oriented organizational and technical systems is considered. The purpose of this article is to use discrete and continuous Markov chains for absorbing states of the system.

III. OUTLINE OF THE MAIN MATERIAL

The scale of compliance rates is considered on the example of project environmental evaluations that meet the criteria (table 1).

Depending on the gradation of compliance states as the degree of project excellence, a model of success criteria is proposed. This model is universal and can be applied to any project and their components that characterize the main aspects of projects. To describe such a model, we use Markov chains with discrete and continuous time [15].

Known examples of the use of the Markov chains to determine the probabilities of states of organizational and technical or social systems are based on the structural and parametric similarity of the originals of these systems by their reflection, i.e. the Markov chains. Using the Markov model, the organizational and technical system of project-oriented management of a machine-building enterprise is presented [7]. It is effective to use the Markov chains to evaluate the quality of educational institutions and to manage communications in advertising projects using the Markov model [8].

A model for assessing the degree of compliance of environmental assessments with quality criteria (Table 1) is presented in the form of an oriented graph. The vertices of the graph correspond to the states of degrees of conformity of ecological estimates to certain criteria, and the arcs to non-zero transition probabilities (Fig. 1).

Here states D1 and D5 are assumed to be absorbing. This means that in the case of transition to states D1 and D5, there is no possibility to pass from them to any other states. For the absorbing state, the transition probabilities obey the conditions $\pi_{ii} = 1, \pi_{ij} = 0$, for i = 1, 5.

TABLE I. DEGREES OF COMPLIANCE OF ENVIRONMENTAL EVALUATIONS WITH THE SUCCESS CRITERIA

Evaluation	Explanation, evaluation criteria	State
A	generally performed well, no important tasks are left unfulfilled	D_1
B	generally satisfactory and complete, there are only minor omissions	D_2
C	satisfactory despite omissions and / or discrepancies	D_3
D	generally unsatisfactory due to significant and substantial omissions and / or discrepancies, even some sections are well executed	D_4
E	extremely unsatisfactory, important tasks are poorly or not done at all	D_5

The inner states Di (i = 2, 3, 4) are irreversible, so that for the number of steps n, $\pi_{ij}(n) > 0$, but $\pi_{ji}(m) = 0 \forall m.$.

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An irreversible state is a state that the process cannot return to after leaving. The process may leave this state but cannot return to it.

Thus, from an irreversible state it is always possible to pass to some other state with a certain number of steps, but at the same time it is impossible to return from this state to the initial state [8 – 9].

It follows from the definition of an irreversible state that if a process comes out of an irreversible state, it can never return to this set. A Markov chain is called absorbing if there is at least one absorbing state among the others.

The presence in the system of absorbing states radically changes the nature of the process.

From the inner states D2 – D4, possible transitions are assumed to be made in the direction of states D1 or D5, with probabilities p and q respectively. It is clear that p + q = 1, and $\pi_{ii} = 0,$ if i = 2, 3, 4.

The transition matrix in this case is:

$$\pi = \begin{pmatrix} \pi_{11} & \pi_{12} & \cdots & \pi_{15} \\ \pi_{21} & \pi_{22} & \cdots & \pi_{25} \\ \vdots & \vdots & & \vdots \\ \pi_{51} & \pi_{52} & \cdots & \pi_{55} \end{pmatrix} = \qquad (1)$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ q & 0 & p & 0 & 0 \\ 0 & q & 0 & p & 0 \\ 0 & 0 & q & 0 & p \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Transition probabilities πik {i = 1,…,n; k = 1,…,n; n=5} can be obtained by expert methods. Transitions between states characterize the level of technological maturity of an organization to some extent. The "delay" probabilities πii add to 1 the sum of the transition probabilities from the i-th state to the other states in one step.

The general solution of the Markov chain represented by the oriented labeled graph in Fig. 1 is obtained on the basis of the transition probability matrix, provided that the system initial state {p1(k), p2(k),…p6(k)} is known:

$$\begin{pmatrix} p_1(k+1) \\ p_2(k+1) \\ p_3(k+1) \\ p_4(k+1) \\ p_5(k+1) \end{pmatrix} = \begin{pmatrix} \pi_{11} & \pi_{12} & \cdots & \pi_{15} \\ \pi_{21} & \pi_{22} & \cdots & \pi_{25} \\ \vdots & \vdots & & \vdots \\ \pi_{51} & \pi_{52} & \cdots & \pi_{55} \end{pmatrix}^T \begin{pmatrix} p_1(k) \\ p_2(k) \\ p_3(k) \\ p_4(k) \\ p_5(k) \end{pmatrix} \qquad (2)$$

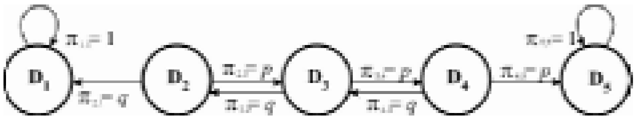

Fig. 1. The marked graph of the success criteria evaluation model

The nature of the probability distribution of the initial states is determined by the conditions of the problem. For example, at an initial moment the system may be in each of the states with equal probability.

The appearance of the transition matrix depends entirely on numbering the states.

First, all the absorbing states are renumbered and then, all the last ones. The data are placed in Table 2.

TABLE II.

Former symbol	D_1	D_2	D_3	D_4	D_5
New symbol	B_1	B_3	B_4	B_5	B_2

Fig. 2 shows the "new" changes in the graph of success criteria evaluation model:

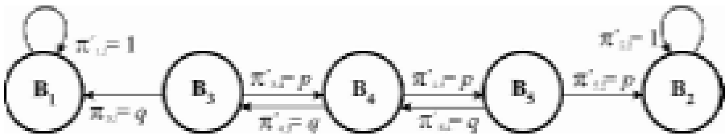

Fig. 2. The "New" marked graph of the success criteria evaluation model

From such an operation, the process in the system does not change, although the transition matrix (1) will be transformed to the form (3):

$$\pi' = \begin{pmatrix} \pi'_{11} & \pi'_{12} & \cdots & \pi'_{15} \\ \pi'_{21} & \pi'_{22} & \cdots & \pi'_{25} \\ \vdots & \vdots & & \vdots \\ \pi'_{51} & \pi'_{52} & \cdots & \pi'_{55} \end{pmatrix} = \qquad (3)$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ q & 0 & 0 & p & 0 \\ 0 & 0 & q & 0 & p \\ 0 & p & 0 & q & 0 \end{pmatrix}.$$

The transition matrix (3) on the submatrix is divided as follows:

$$\boldsymbol{\pi'} = \left|\begin{array}{cc|ccc} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ \hline q & 0 & 0 & p & 0 \\ 0 & 0 & q & 0 & p \\ 0 & p & 0 & q & 0 \end{array}\right| = \left|\begin{array}{c|c} \boldsymbol{I} & \boldsymbol{O} \\ \hline \boldsymbol{R} & \boldsymbol{Q} \end{array}\right| \qquad (4).$$

If n is the dimension of the system, r is the number of absorbing states, then n-r is the number of irreversible states. Submatrices have the following dimensions:

I = I(r×r), I - a single matrix whose order is determined by the number of absorbing states in the system;

O = O(r×n-r), O – a zero matrix;

R = R(n-r×r), R – consists of elements that characterize the transition from irreversible states to absorbing;

Q = Q(n-r×n-r), Q – a matrix that describes the behavior of a system or process in a set of irreversible states before the transition to absorbing states.

In this case n = 5, r = 2, hence the dimensions of the matrices are respectively the following:

I = I(2×2), O = O(2×3), R = R(3×2), Q = Q(3×3).

The representation of the transition matrix in the form (4) is called canonical.

The main feature of absorbing states is that as the number of steps increases ($n \to \infty$), the probability of the process or system getting into the absorbing state is 1. As n increases, the elements of the submatrix Q tend to 0 and the ones of the submatrice R to 1.

The nature of changing elements of the submatrix Q with increasing n is associated with the determination of important quantitative characteristics of the absorbing chains:

1) the probability of reaching the absorbing state of any given;
2) the average value of the number of steps required to achieve an absorbing state;
3) the average time spent by the system in each of the irreversible states before the system enters the absorbing state.

The number n_j of the process getting into an irreversible state x_j is calculated. The number n_j multiplied by the unit of time characterizes the time of being in this state. The number n_j is a random variable and its characteristics depend on the submatrix of the transition probabilities Q and on the initial state. The mean value n_j is identified by $\left(\overline{n_j}\right)_i$, where $\overline{n_j}$ means the operation of averaging over the set, and the index indicates that the average value is calculated for the i-th learning state. The value $\left(\overline{n_j}\right)_i$ includes an addend that reflects the fact that the process is in its original state. Analytically consider this with the Kronecker symbol:

$$\delta_{ij} = \begin{cases} 1, & if \;\; i = j, \\ 0, & if \;\; i \neq j. \end{cases}$$

After the first step, the process with probability π_{ik} will likely go to the state D_k belonging to the set T of all irreversible states. Added all over k, the following is obtained:

$$\left(\overline{n_j}\right)_i = \delta_{ij} + \sum_{k \in T} \pi_{ik} \left(\overline{n_j}\right)_k . \qquad (5)$$

Based on the formula (5), and, taking into account the rules of addition and multiplication of matrices, the matrix relation is obtained:

$$N = N\left(\left(\overline{n_j}\right)_i\right) = (I - Q)^{-1} \qquad (6)$$

Using the fundamental matrix N, which is determined by relation (6), it is possible to calculate the various characteristics of the process.

Each element of the matrix N denotes the average number of times the process enters the given irreversible state, depending on the initial state. The elements of the main diagonal are larger than 1.

We find the fundamental matrix for the given absorbing Markov chain.

$$Q = \begin{pmatrix} 0 & p & 0 \\ q & 0 & p \\ 0 & q & 0 \end{pmatrix}, \qquad (7)$$

$$I - Q = \begin{pmatrix} 1 & -p & 0 \\ -q & 1 & -p \\ 0 & -q & 1 \end{pmatrix}.$$

$$N = (I - Q)^{-1} = \qquad (8)$$

$$= \frac{1}{1 - 2pq} \begin{pmatrix} 1 - pq & p & p^2 \\ q & 1 & p \\ q^2 & q & 1 - pq \end{pmatrix}.$$

If it is set, that p = 0.25 a q respectively q = 0.75, then the transition matrix (1) will have the following form:

$$\pi = \begin{pmatrix} \pi_{11} & \pi_{12} & \cdots & \pi_{15} \\ \pi_{21} & \pi_{22} & \cdots & \pi_{25} \\ \vdots & \vdots & & \vdots \\ \pi_{51} & \pi_{52} & \cdots & \pi_{55} \end{pmatrix} = \qquad (9)$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0{,}75 & 0 & 0{,}25 & 0 & 0 \\ 0 & 0{,}75 & 0 & 0{,}25 & 0 \\ 0 & 0 & 0{,}75 & 0 & 0{,}25 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

The fundamental matrix is

$$\mathbf{N} = \begin{pmatrix} 13/10 & 2/5 & 1/10 \\ 6/5 & 8/5 & 2/5 \\ 9/10 & 6/5 & 13/10 \end{pmatrix}. \qquad (10)$$

If the elements of the second row of the matrix (10) are considered, it will be clear that if the process starts from a state B_3, then taking into account the equality of the 1 of the initial state, the process in this state will average 8/5 units of time.

From the initial moment, the process will hold in the state B_2 6/5 units of time, and in the state B_4 only 2/5.

The values of the elements of the fundamental matrix last rows are interpreted similarly.

IV. Conclusions

Due to the homogeneity of the Markov chain, any state in which the system is detected at a given time can be selected as the initial state. Therefore, the fundamental matrix gives the same prediction for the future, regardless of the absolute value of the time elapsed from the initial moment. This property of the fundamental matrix illustrates the Markov property of the process, characterizing it as a process without aftereffect, i.e. in the known present, the future is independent of the past.