Cellular Automata as Discrete Dynamical Systems

Workshop “Dynamical systems and stability”
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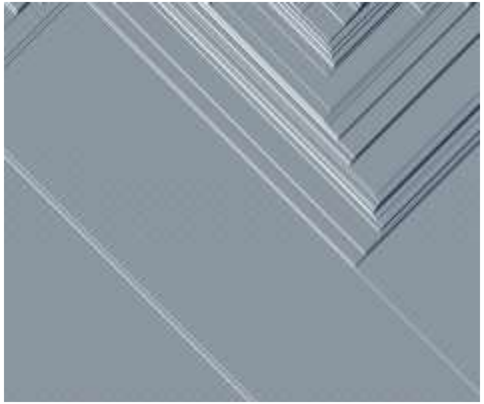
Objective

1. Representation a dynamical system that set CA as difference equations
2. Generalization CA for possibility of each cell to take multiplicity of states

Application of Cellular Automata (CA)

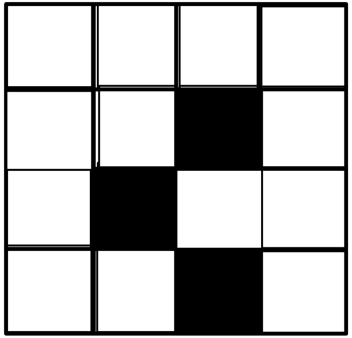
- Randomizing of number sequences (rule 30 of Wolfram Code)
- Localization of the epidemic spreading
- Modeling of traffic flows (rule 184 of Wolfram code)
- Modeling processes in Chemistry (Belousov-Zhabotinsky reaction), Physics (turbulence), Social Sciences, etc.



Pic.1 Rule 184, traffic flows model

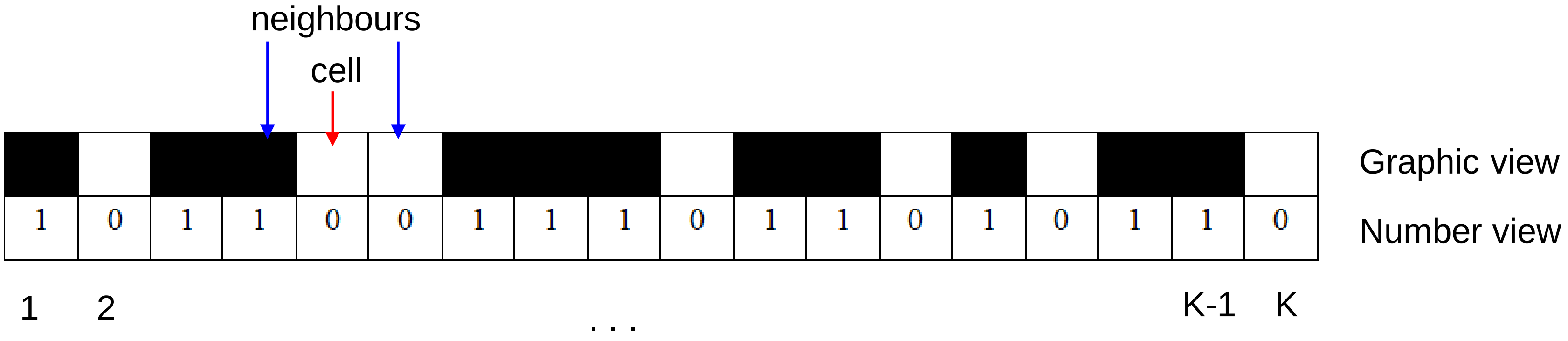
Pic.2 Belousov-Zhabotinsky reaction

What Is The Cellular Automata?



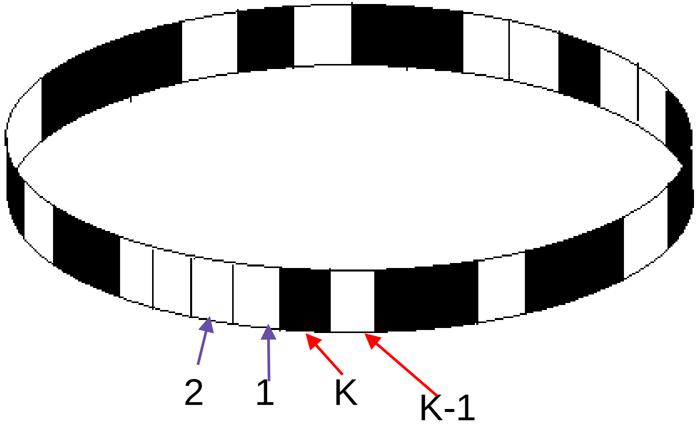
- CA is a discrete dynamical system, has form of lattice
- Every it’s cell takes one state from the finite set
- There are some rules for changing cell’s state on the next step

The simplest linear CA



The state of a cell on the next step depends on current state of this cell and its neighbours states

Besides, neighbours for first cell are cell 2 and cell K; neighbours for cell K are cell K-1 and cell 1



Discrete dynamical system

Let’s assume that the map of some set into itself is given as:

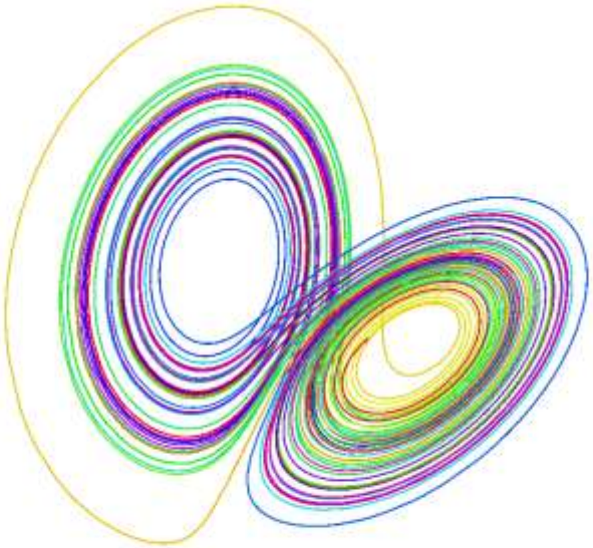
$$x_{n+1} = f(x_n), f : A \to A, A \subset R^m$$

A - convex invariant set

A fixed point is any element $\eta \in A$ such that:

$$f(\eta) = \eta$$

Periodic orbits



- The orbit of a point η_1 is the set

$$O(\eta_1) = \{\eta_1, f(\eta_1), f_2(\eta_1), \dots\}$$

where

$$f_2(\eta_1) = f(f(\eta_1)), \qquad f_{K+1}(\eta_1) = f(f_K(\eta_1)), \qquad K = 2,3, \dots$$

For T-periodic point the set

$$O(\eta_1) = \{\eta_1, f(\eta_1), \dots, f_T(\eta_1)\} = \{\eta_1, \dots, \eta_T\}$$

$(\eta_i \neq \eta_j$ if $i \neq j, f_{T+1}(\eta_1) = \eta_1)$ is a periodic orbit

Multipliers

One of criteria of the local stability of periodic points:

all eigenvalues of the Jacobi matrix $f'(\eta_1)\dots f'(\eta_T)$ are in the central unit disc in the complex plane

Eigenvalues are roots of the characteristic equation

$$\det(\mu I - \prod_{j=1}^{T} f'(\eta_j)) = 0$$

Let's denote them $\{\mu_1, \dots, \mu_m\}$, - multipliers

Assume $\mu_j \in M, j = 1, \dots, m$ - region of multipliers localization

How to find a fixed point?

And how to find periodic point?

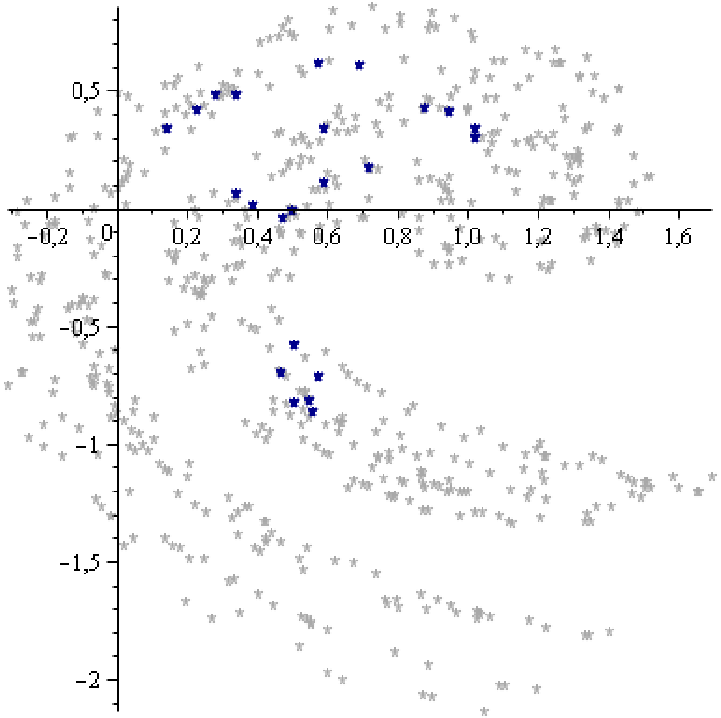
The problem though is: how one can check the convergence of the recurrent sequence

$$x_{n+1} = f(x_n)$$

if nothing is known about the multipliers?

What if some multipliers are outside the central unit disk?

Control to find cycles?



There are two basic methods:

- Newton-Raphson
- Average damping

Let’s consider the Average damping method:

$$x_{n+1} = F(x_n, u_n)$$

where u_n is a control as a function of the values for the vector of recurrent sequence computed on the previous step

Control

Consider the system

$$Y_{n+1} = F(Y_n)$$

The system with control is

$$X_{n+1} = \tilde{F}(X_n, U_n)$$

where $F(X_n) = \tilde{F}(X_n, 0)$

and periodic orbit become locally-asymptotically stable

Delayed Feedback Control (DFC)

Linear DFC (K.Pyragas)	Nonlinear DFC (M.Viera, A.Lichtenberg)	Multilinear DFC (O.Morgul)
$u_n = \epsilon(x_{n-T+1} - x_{n-T})$	$u_n = \epsilon(f(x_{n-T+1}) - f(x_{n-T}))$	$u_n = \epsilon(x_{n-T+1} - f(x_{n-T}))$

Generalized Average Damping Control

Using of generalized nonlinear and multilinear controls

$$u_n = -(1-\gamma)\sum_{j=1}^{N}\epsilon_j\left(f(x_{n-jT+T}) - f(x_{n-jT})\right) - \gamma\sum_{j=1}^{N}\delta_j\left(f(x_{n-jT+T}) - x_{n-jT+1}\right)$$

$$\gamma \in [0,1]$$

Controlled system

Thus, the recurrent equation has form

$$x_{n+1} = (1-\gamma) f\left(\sum_{j=1}^{N} a_j x_{n-jT+T}\right) + \gamma \sum_{j=1}^{N} b_j x_{n-jT+1}$$

$$\sum_{j=1}^{N} a_j = 1, \sum_{j=1}^{N} b_j = 1$$

N is a prehistory

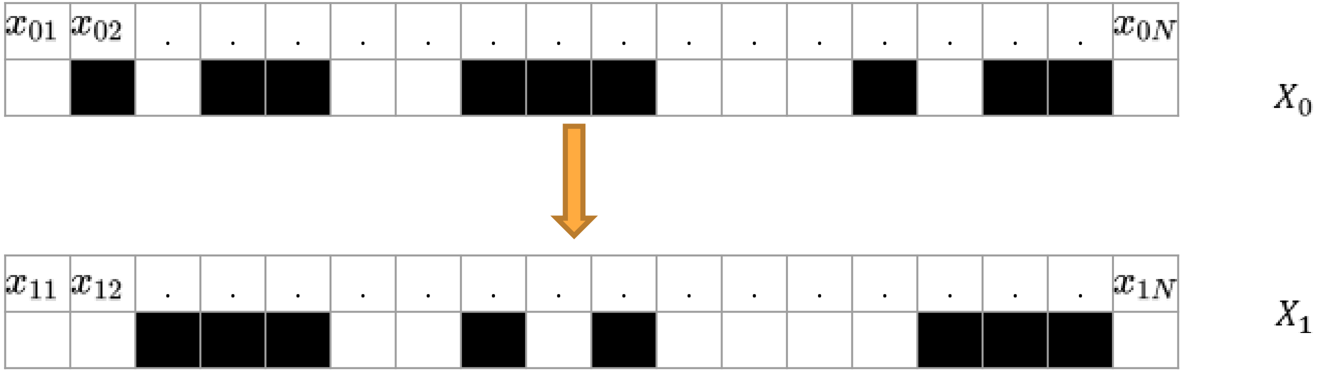
a_j, b_j and γ chosen such, that N would be minimal

There is an algorithm for determining a_j

Dmitrishin D., Stokolos A., Skrynnik I., F. E. (2017) Generalization of Nonlinear Control for Nonlinear Discrete Systems, Available at: https://arxiv.org/pdf/1709.10410.pdf

From CA to Dynamical System

Let's assume X_0 as vector of CA states



Nonlinear Discrete Equations as System Diffusion-Reaction

Diffusion equation:

$$\begin{cases} y_i = \sum_{s=-r}^{r} \delta_s x_{i+s} \\ i = \overline{1, K} \end{cases}$$

δ_s - positive integer weight coefficients

r - quantity of neighbors

K - size of CA

The term “Diffusion-Reaction” was taken from work

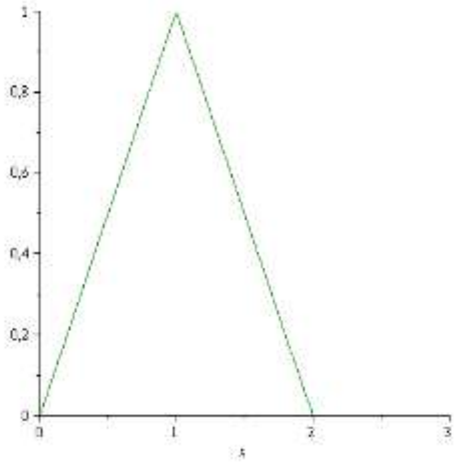
Scalise D., Schulman R., (2015) Emulating cellular automata in chemical reaction–diffusion networks, *Natural Computing,* Volume 15, pp. 197–214, doi: 10.1007/s11047-015-9503-8

Tent functions

Let's denote

$$p = \sum_{s=-r}^{r} \delta_s + 1$$

Consider the set of Tent functions



$$\varphi(j) = \begin{cases} x - j + 1, & j - 1 < x < j \\ j - x + 1, & j \le x \le j + 1 \end{cases}, \quad j = \overline{1, p}$$

$$\varphi(0) = \begin{cases} x - p, & p < x < p + 1 \\ 1 - x, & 0 \le x \le 1 \end{cases}$$

Reaction Equation

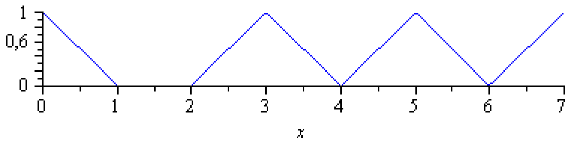
Let's denote the Diffusion equation as:

$$Y_{n+1} = DX_n$$

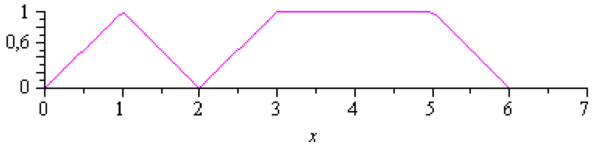
Thus, the Reaction equation is

$$X_{n+1} = \Phi(DX_n)$$

Where $\Phi = \begin{pmatrix} \varphi_1 \\ \vdots \\ \varphi_K \end{pmatrix}$



$$X_{n+1} = \varphi(0) + \varphi(3) + \varphi(5)$$



$$X_{n+1} = \varphi(1) + \varphi(3) + \varphi(4) + \varphi(5)$$

Reaction equation represent the sum of Tent functions

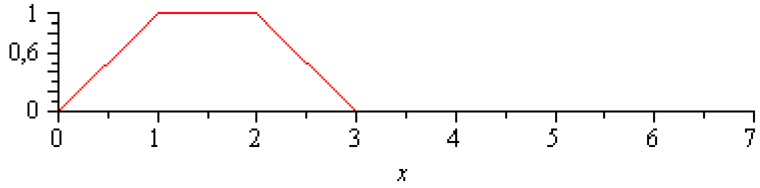
Example

Let's turn the 30th rule of Wolfram Code to Dynamical System

111	110	101	100	011	010	001	000	Current state of cell / Current state of cell's neighbors
0	0	0	1	1	1	1	0	State of cell on the next step
4	3	3	2	2	1	1	0	$\{\delta_1, \delta_2, \delta_3\} = \{2, 1, 1\}$

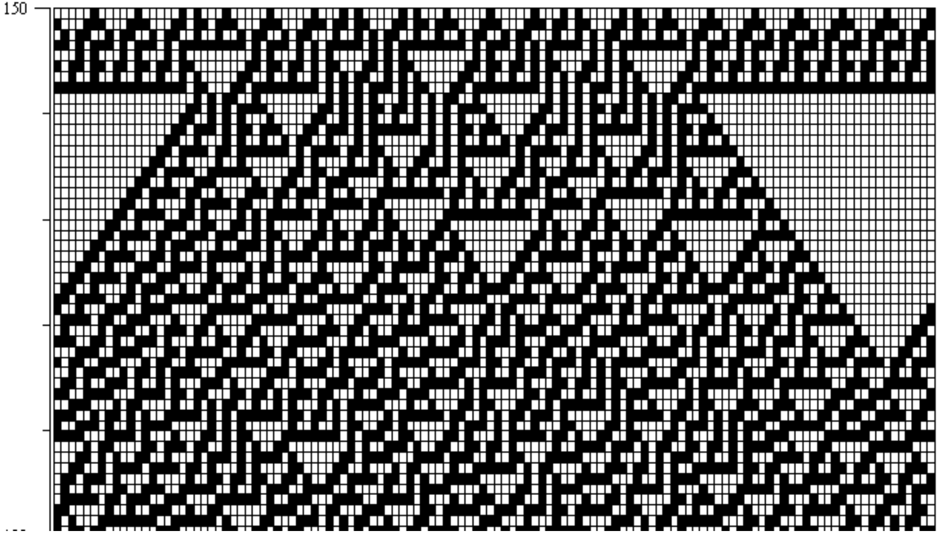
So the function of reaction for 30th rule is

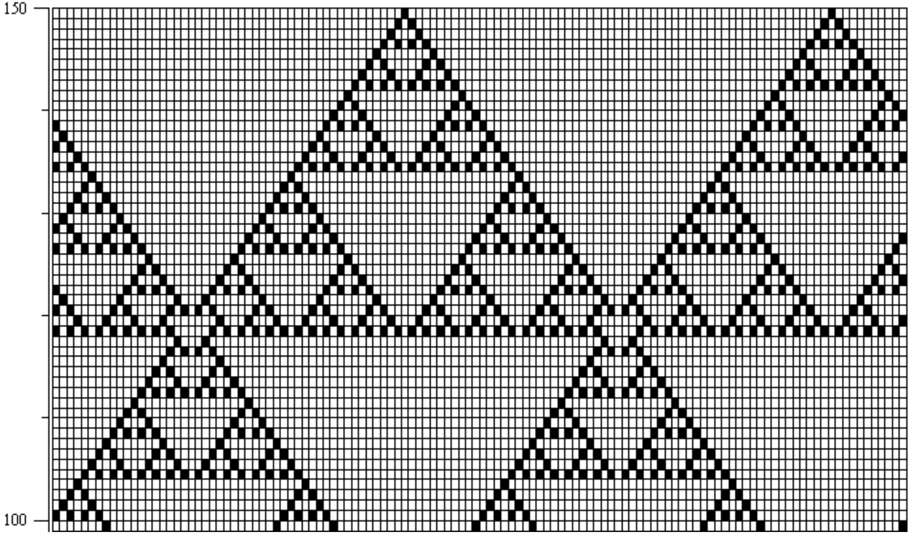
$$F(x) = \varphi(1) + \varphi(2)$$



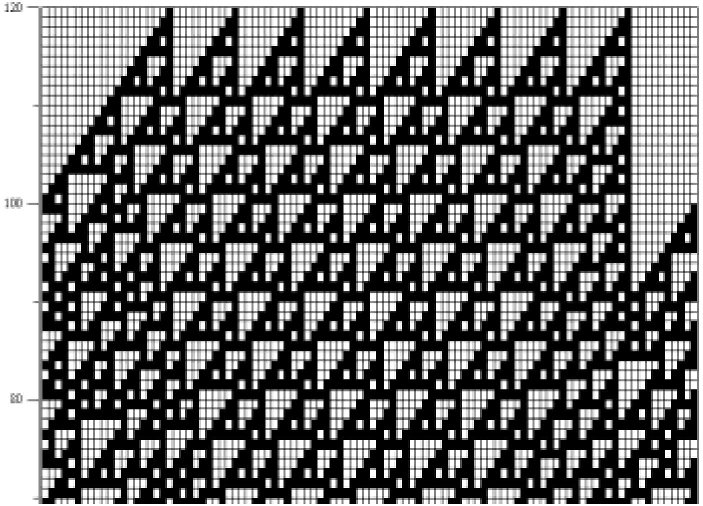
The tent map of 30th Rule

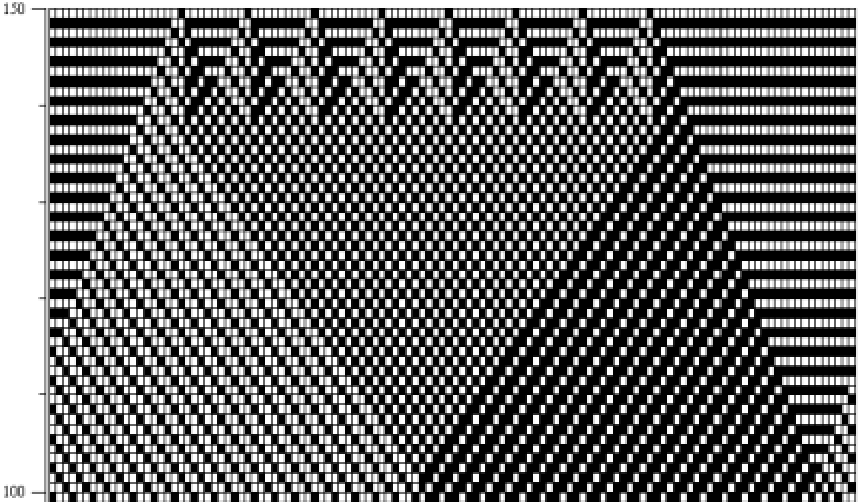
Rule 30 and 90



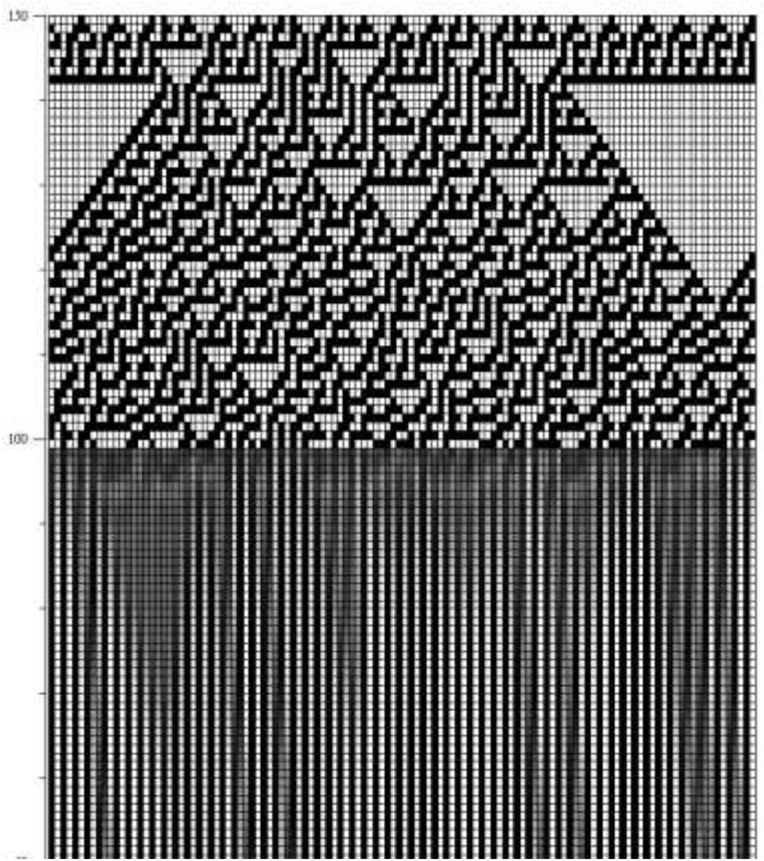


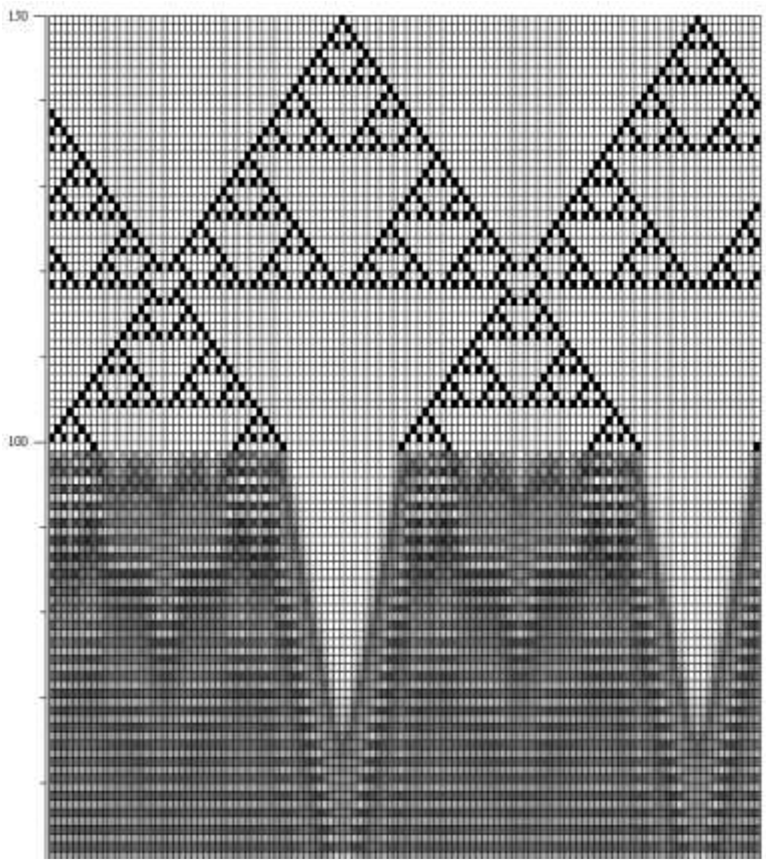
Rule 110 and 57



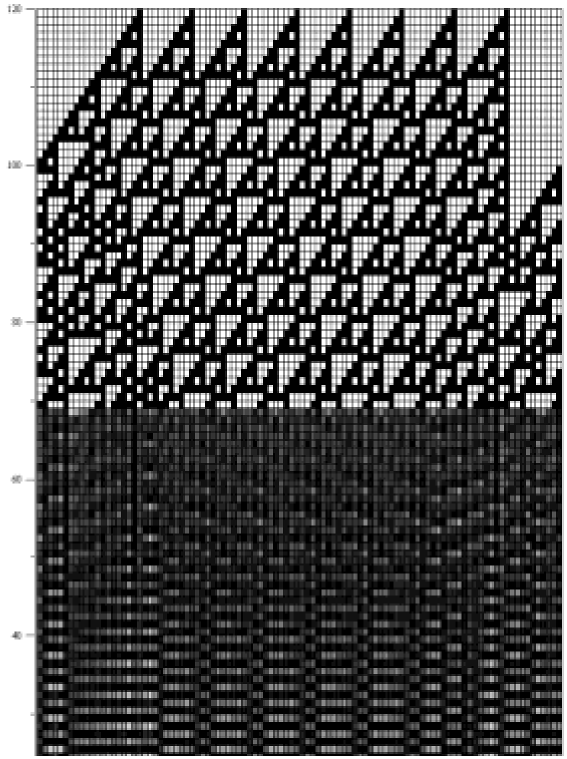


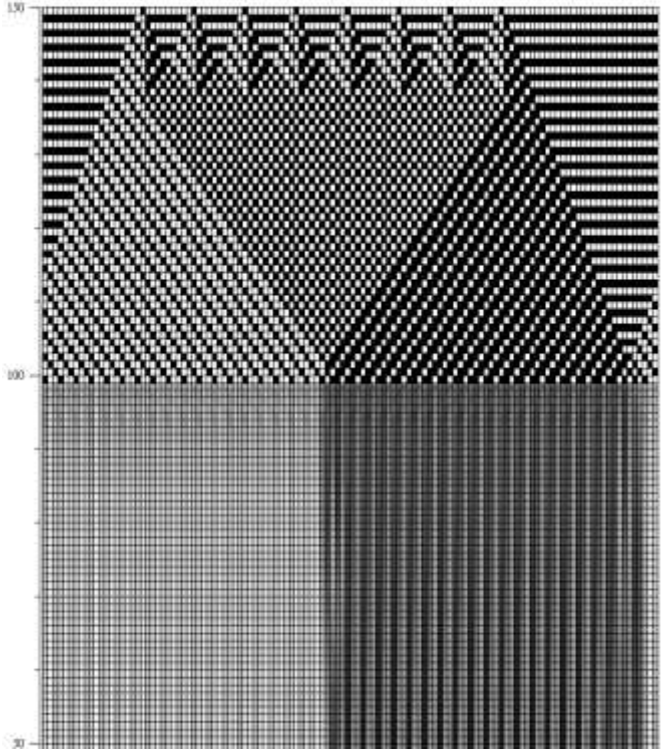
Rule 30 and 90 with control T=1



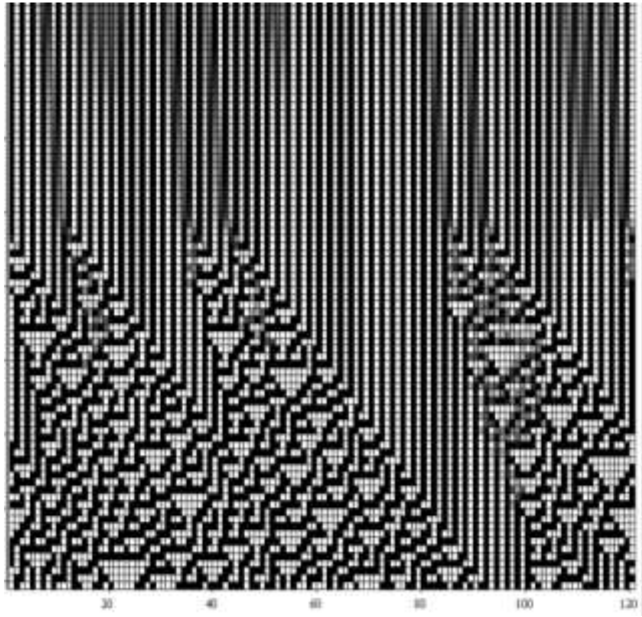


Rule 110 and 57 with control T=1

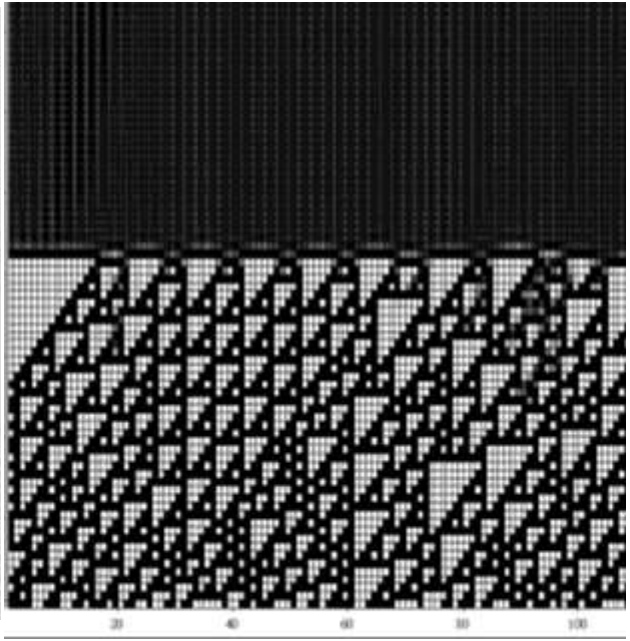




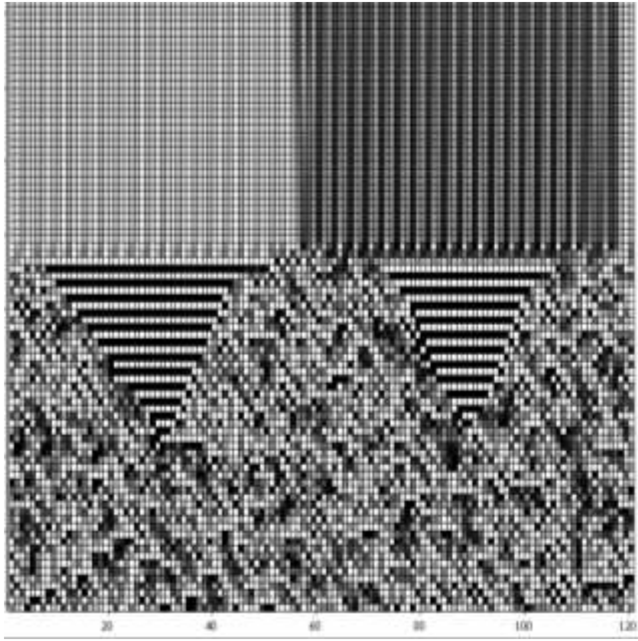
Some interesting behavior of CA after control's switching off



Rule 30



Rule 110



Rule 57

Summarize

- The instrument for research CA exists
- We can apply control for CA systems

Opened problems:

- Applying presented approach for CA with another dimension
- Understanding of behavior of CA under control
- Existing of periodic or quasiperiodic structures in CA
- Control concrete processes that are modeled by CA

Thank you for your attention!

“Yet you balanced an eel on the end of your nose –
What made you so awfully clever?”

John Tenniel’s illustrations to Alice’s Adventures in Wonderland (1865). Chapter 5.

Literature

1. Dmitrishin D., Stokolos A., Skrynnik I., Franzheva E. (2017) Generalization of Nonlinear Control for Nonlinear Discrete Systems, Available at: https://arxiv.org/pdf/1709.10410.pdf
2. Scalise D., Schulman R., (2015) Emulating cellular automata in chemical reaction–diffusion networks, *Natural Computing*, Volume 15, pp. 197–214, doi: 10.1007/s11047-015-9503-8