Journal of Communications Technology and Electronics, Vol. 50, No. 12, 2005, pp. 1367–1372. Translated from Radiotekhnika i Elektronika, Vol. 50, No. 12, 2005, pp. 1476–1481.

THEORY AND METHODS OF SIGNAL PROCESSING

Modified Signal-Processing Algorithms Based on the Hartley Transform

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Received August 11, 2004

Abstract—Modified algorithms based on the Hartley transform (HT) are developed for solution of such important problems of digital signal processing as the calculation of cyclic convolutions and cross-correlation and autocorrelation functions. Data transmission based on orthogonal frequency division multiplex (OFDM) is considered. The proposed data-transmission system employs the HT as an algorithm forming orthogonal channel signals. The use of the developed algorithms simplifies conventional computational procedures and lowers the amount of required RAM. The results obtained can be applied for signal filtering and designing such advanced network standards as HiperLAN/2 and IEEE802.11a.

The most widespread operations of digital signal processing are the calculation of digital convolutions, correlation functions, and spectra of various signals [1, 2]. In the recent papers, the Hartley transform (HT) has been applied widely as an alternative to the Fourier transform (FT).

The HT was proposed in [3] and, later, covered extensively in the literature on digital signal processing [4–8]. The HT is superior to the FT, in particular, in calculation of signal spectra because the HT allows replacement of the FT complex basis by the HT real basis that retains, simultaneously, number N of the degrees of freedom of the transformation [7]. In addition, the direct and inverse HTs are implemented in the same algorithm. For an arbitrary function that is continuous in time and frequency, the direct and inverse HTs are defined as follows:

$$H(f) = \int_{-\infty}^{\infty} h(t)\mathrm{cas}(2\pi f t)dt,$$

$$h(t) = \int_{-\infty}^{\infty} H(f)\mathrm{cas}(2\pi f t)df,$$

where $\mathrm{cas}(z) = \cos(z) + \sin(z)$ and z is an arbitrary value of the argument.

Solution of digital-signal-processing problems necessitates discretization in time or frequency or simultaneous discretization in both variables. Thus, frequency discretization yields the direct and inverse frequency-discrete HTs

$$H(k) = \int_{0}^{T} h(t)\mathrm{cas}(2\pi f_k t)dt, \qquad (1)$$

$$h(t) = (1/T)\sum_{k=-\infty}^{\infty} H(k)\mathrm{cas}(2\pi f_k t), \qquad (2)$$

respectively, where T is the duration of a signal.

Simultaneous time and frequency discretization yields the direct and inverse discrete HT (DHT)

$$H(k) = N^{-1}\sum_{n=0}^{N-1} h(n)\mathrm{cas}\left(\frac{2\pi nk}{N}\right), \qquad (3)$$

$$h(n) = \sum_{k=0}^{N-1} H(k)\mathrm{cas}\left(\frac{2\pi nk}{N}\right). \qquad (4)$$

Expressions (3) and (4) imply that both the direct and inverse DHTs have the same kernels of transformation and differ only by the normalization factor. Therefore, these transforms can be calculated using one and the same fast algorithm: the fast Hartley transform (FHT) [4, 7].

The Fourier and Hartley transforms are related by the following simple formulas:

$$\mathrm{Re}F(k) = (1/2)[H(k) + H(N-k)], \quad k = 0, 1, \ldots, N-1, \qquad (5)$$

$$\mathrm{Im}F(k) = 1/2[H(k) - H(N-k)], \qquad (6)$$

where $F(k)$ and $H(k)$ are the FT and HT, respectively, of an arbitrary discrete function.

For arrays of real data, the following inverse relationship holds:

$$H(k) = (1/2)[F(k) + F(N-k)]. \tag{7}$$

Obviously, like the FT, the HT has an orthogonal harmonic decomposition basis. Equations (5)–(7) describe the linear relationship between Fourier and Hartley spectra and, hence, indicate the possibility of using the HT for calculation of a spectrum from its HT. The properties of the HT resemble those of the FT [7]. In addition, the orthogonality of the Hartley basis provides, in certain cases, for signal processing based on the HT instead of the FT.

In this paper, modified HT-based algorithms are proposed for calculation of digital cyclic convolutions, correlation functions, and autocorrelation functions. These algorithms are simpler and require less RAM as compared to conventional procedures. The developed algorithms employ two transformations instead of three different HTs. Therefore, the computational cost and required RAM are reduced by one-third and, as a result, the CPU time necessary for computation of a convolution is reduced. We suggest applying the HT instead of the FT in orthogonal frequency division multiplex (OFDM) for simplification of a transceiver. The gain in the computational efficiency is demonstrated.

First, let us consider calculation of the cyclic digital convolution of two functions $h(n)$ and $q(n)$ defined by the expression

$$h(n) * q(n) = N^{-1}\sum_{m=0}^{N-1} h(m)q(n-m), \tag{8}$$

where * is the sign of a convolution of two digital signals.

A lot of computational methods based on formula (8) have been proposed in the literature [1, 2, 9]. They differ in both computational complexity and hardware implementation.

In [2, 7], it is proposed to calculate a cyclic convolution using the frequency-domain Hartley transform defined as

$$\begin{aligned} h(n) * q(n) &\Leftrightarrow H(k)Q_s(k) + H(N-k)Q_c(k) \\ &= Q(k)H_s(k) + Q(N-k)H_c(k), \end{aligned} \tag{9}$$

where $H(k)$ and $Q(k)$ are the Hartley spectra of functions $h(n)$ and $q(n)$, $n = 0, 1, \ldots, N-1$, and $\Leftrightarrow$ is the sign of correspondence.

The following notation is employed in formula (9):

$$H_s(k) = N^{-1}\sum_{n=0}^{N-1} h(n)\sin\left(\frac{2\pi nk}{N}\right), \tag{10}$$

$$H_c(k) = N^{-1}\sum_{n=0}^{N-1} h(n)\cos\left(\frac{2\pi nk}{N}\right), \tag{11}$$

$$Q_s(k) = N^{-1}\sum_{n=0}^{N-1} q(n)\sin\left(\frac{2\pi nk}{N}\right), \tag{12}$$

$$Q_c(k) = N^{-1}\sum_{n=0}^{N-1} q(n)\cos\left(\frac{2\pi nk}{N}\right). \tag{13}$$

It follows from expressions (9)–(13) that the algorithm developed for calculation of a cyclic convolution necessitates computation of three transforms: $H(k)$, $Q_c(k)$, and $Q_s(k)$.

Let us show that the number of transforms can be reduced to two. We apply the trigonometric identities

$$2\cos(z) = \text{cas}(z) + \text{cas}(-z), \tag{14}$$

and

$$2\sin(z) = \text{cas}(z) - \text{cas}(-z), \tag{15}$$

to synthesize the algorithm.

Having substituted expressions (14) and (15) into Eq. (10) and taking into account the periodicity of the sine and cosine functions and the obvious equalities $H(k) = Y(N-k)$ and $Y(k) = H(N-k)$, which can easily be proved by inverting the order of summation ($k' = N-k$), we obtain

$$\begin{aligned} H_s(k) &= \frac{N^{-1}}{2}\sum_{n=0}^{N-1} h(n)\text{cas}\left(\frac{2\pi nk}{N}\right) \\ &\quad - \frac{N^{-1}}{2}\sum_{n=0}^{N-1} h(n)\text{cas}\left(-\frac{2\pi nk}{N}\right) \\ &= \frac{1}{2}[H(k) - Y(k)] = \frac{1}{2}[H(k) - H(N-k)], \end{aligned} \tag{16}$$

where $k = 0, 1, 2, \ldots, N-1$ and functions $H(k)$ and $Y(k)$ are defined as

$$H(k) = N^{-1}\sum_{n=0}^{N-1} h(n)\text{cas}\left(\frac{2\pi nk}{N}\right), \tag{17}$$

$$Y(k) = N^{-1}\sum_{n=0}^{N-1} h(n)\text{cas}\left(-\frac{2\pi nk}{N}\right). \tag{18}$$

Applying successively a similar substitution in Eqs. (11)–(13), we obtain

$$H_c(k) = \frac{N^{-1}}{2}\sum_{n=0}^{N-1} h(n)\mathrm{cas}\left(\frac{2\pi nk}{N}\right) + \frac{N^{-1}}{2}\sum_{n=0}^{N-1} h(n)\mathrm{cas}\left(-\frac{2\pi nk}{N}\right) \tag{19}$$
$$= \frac{1}{2}[H(k) + Y(k)] = \frac{1}{2}[H(k) + H(N-k)],$$

$$Q_s(k) = \frac{N^{-1}}{2}\sum_{n=0}^{N-1} q(n)\mathrm{cas}\left(\frac{2\pi nk}{N}\right) - \frac{N^{-1}}{2}\sum_{n=0}^{N-1} q(n)\mathrm{cas}\left(-\frac{2\pi nk}{N}\right) \tag{20}$$
$$= \frac{1}{2}[Q(k) + X(k)] = \frac{1}{2}[Q(k) - Q(N-k)],$$

$$Q_c(k) = \frac{N^{-1}}{2}\sum_{n=0}^{N-1} q(n)\mathrm{cas}\left(\frac{2\pi nk}{N}\right) + \frac{N^{-1}}{2}\sum_{n=0}^{N-1} q(n)\mathrm{cas}\left(-\frac{2\pi nk}{N}\right) \tag{21}$$
$$= \frac{1}{2}[Q(k) + X(k)] = \frac{1}{2}[Q(k) + Q(N-k)],$$

where

$$Q(k) = \frac{N^{-1}}{2}\sum_{n=0}^{N-1} q(n)\mathrm{cas}\left(\frac{2\pi nk}{N}\right), \tag{22}$$

$$X(k) = \frac{N^{-1}}{2}\sum_{n=0}^{N-1} q(n)\mathrm{cas}\left(-\frac{2\pi nk}{N}\right). \tag{23}$$

In expressions (19)–(21), it is taken into account that $Y(k) = H(N-k)$ and $X(k) = Q(N-k)$.

With allowance for (16)–(23), expression (9) takes the final form

$$h(n) * q(n) \Leftrightarrow \frac{1}{2}\{H(k)[Q(k) + Q(N-k)] + H(N-k)[Q(k) - Q(N-k)]\}. \tag{24}$$

It follows from (24) that, in order to calculate the digital convolution of two functions $h(n)$ and $q(n)$, it is sufficient to determine Hartley spectra $H(k)$ and $Q(k)$ of these functions and calculate their linear combination. If the convolution is calculated according to algorithm (9), it is necessary to determine three discrete functions $H(k)$, $Q_s(k)$, and $Q_c(k)$ instead of two as required by algorithm (24). Thus, the use of algorithm (24) reduces both the computational work and the required RAM by a third because it is necessary to store two rather than three arrays.

Let us apply the FHT to calculate a cross-correlation function (CCF). Taking into account the definition of the CCF of discrete signals $h(n)$ and $q(n)$ from [2, 7], we obtain

$$h(n) ** q(n) \Leftrightarrow H(N-k)Q_c(k) + H(k)Q_s(k) = Q(N-k)H_c(k) + Q(k)H_s(k), \tag{25}$$

where ** is the sign of either correlation or autocorrelation.

By analogy to expressions (9)–(23), we find

$$Q_c(k) = \frac{N^{-1}}{2}\sum_{n=0}^{N-1} q(n)\mathrm{cas}\left(\frac{2\pi nk}{N}\right) + \frac{N^{-1}}{2}\sum_{n=0}^{N-1} q(n)\mathrm{cas}\left(-\frac{2\pi nk}{N}\right) \tag{26}$$
$$= \frac{1}{2}[Q(k) + X(k)] = \frac{1}{2}[Q(k) + Q(N-k)],$$

$$Q_s(k) = \frac{N^{-1}}{2}\sum_{n=0}^{N-1} q(n)\mathrm{cas}\left(\frac{2\pi nk}{N}\right) - \frac{N^{-1}}{2}\sum_{n=0}^{N-1} q(n)\mathrm{cas}\left(-\frac{2\pi nk}{N}\right) \tag{27}$$
$$= \frac{1}{2}[Q(k) - X(k)] = \frac{1}{2}[Q(k) - Q(N-k)],$$

where $k = 0, 1, 2, \ldots, N-1$ and $X(k) = Q(N-k)$.

The final expression for the CCF has the form

$$h(n) ** q(n) \Leftrightarrow \frac{1}{2}\{H(N-k)[Q(k) + Q(N-k)] + H(k)[Q(k) - Q(N-k)]\}. \tag{28}$$

It follows from expressions (25) and (28) that calculation of a CCF also necessitates two rather than three transforms. Therefore, a similar reduction in both the number of operations and the required RAM is guaranteed.

It is obvious that, by analogy to (28), the expression for the autocorrelation function (AF) of discrete signal $h(n)$ has the following form in the spectral domain:

$$h(n) ** h(n) \Leftrightarrow \frac{1}{2}\{H_1(N-k)[H_2(k)+H_2(N-k)] + H_1(k)[H_2(k)-H_2(N-k)]\},$$

where $H_1(k)$ and $H_2(k)$ are the Hartley spectra of function $h(n)$ and its shift $h(n + k)$, respectively. All aforementioned gains in the computational efficiency and RAM space are retained.

Since the Hartley basis is orthogonal, it may be applied sometimes instead of the Fourier basis. Let us consider, for example, the possibility of employing the HT for OFDM. With this kind of modulation, which is applied widely in modern communications systems for data transmission, it is possible to perform separate processing of signals represented by a sufficiently large number of orthogonal carriers in the frequency domain [10–12]. The key OFDM operation is the inverse FT (IFT) in the transmitter and the direct FT (DFT) in the receiver. If the input signal is represented as discrete complex data sequence $c(f_n)$ consisting of frequency-domain samples, the corresponding signal in the time domain is determined as

$$s(t) = (1/T_s)\sum_{n=0}^{N-1} c(f_n)\exp(j2\pi f_n t), \quad (29)$$

where $c(f_n) = \sum_i d(n, i)g(t - iT_s)$ are the constellation code symbols of an m-ary ($m = 2, 4, 8, 16, \ldots$) digital modulation either phase shift keying or quadrature amplitude one [13], the function $g(t) = \mathrm{rect}(t/T_s)$ describes the rectangular form of symbol transmission of duration T_s, $f_n = n/T_s$ is the frequency corresponding to the transmitter with index n, and $iT_s \le t \le (i + 1)T_s$.

In expression (29), signal $s(t)$ is equivalent to a frequency-splitting signal (f_n denotes discrete frequencies involved in frequency splitting) that is phase- or amplitude-and-phase-shift keyed in each channel. It is evident that (29) is the IFT of sequence $c(f_n)$. In the receiver, the signal is recovered via the DFT on each temporal interval $[iT_s, (i + 1)T_s]$:

$$\hat{c}(f_n) = \int_{iT_s}^{(i+1)T_s} r(t)\exp(-j2\pi f_n t)dt, \quad (30)$$

where $r(t) = s(t) * h_c + \xi(t)$ is the convolution of the input signal and the pulse response of the communication channel mixed with an additive Gaussian white noise.

In practice, most OFDM systems use the inverse and direct discrete FTs (IDFT and DFT) or the inverse and direct fast FTs (IFFT and FFT) [1, 12]. In this case, the temporal interval $[iT_s, (i + 1)T_s]$ is split into N equal segments and expressions (29) and (30) take the forms

$$s(k) = (1/N)\sum_{n=0}^{N-1} c(n)\exp(j2\pi nk/N) = \mathrm{IDFT}\{c(0), c(1), \ldots, c(N-1)\}, \quad (31)$$

$$\hat{c}(n) = \sum_{k=0}^{N-1} r(k)\exp(-j2\pi nk/N) = \mathrm{DFT}\{r(0), r(1), \ldots, r(N-1)\}. \quad (32)$$

Although the idea of OFDM based on the IFFT and FFT is rather simple, its implementation is impeded by certain factors. Since OFDM exhibits its advantages at N of about several thousands, computational algorithms (31) and (32), which involve complex arithmetical operations, are rather complex despite that the FFT algorithm requires only about $N\log_2 N$ arithmetical operations [1].

Permutation of input data is another procedure that is difficult to implement. Permutations are necessary for the following reason. Even if an original input signal is real, its FT is complex. In practice, it is more convenient to deal with real data arrays. These are formed via intricate permutations of the original array. The corresponding procedure can be described as follows [1]. Let $\{a_m\}$ and $\{b_m\}$ be two real sequences (of length M) of a complex signal of a discrete argument. A new sequence of the length $N = 2M$ is determined such that

$$\begin{aligned} c(n) &= a(m) + jb(m), \\ n = N/2 - m + 1, &\quad m = 1, 2, \ldots, N/2, \\ c(n) &= a(m) - jb(m), \\ n = N/2 + m, &\quad m = 1, 2, \ldots, N/2. \end{aligned} \quad (33)$$

The signal formed exhibits complex-conjugate symmetry. Therefore, upon applying the IFFT, the signal is real owing to the well-known properties of the FT [1]. However, the permutation procedure necessitates a considerable additional amount of RAM. In addition, implementation of OFDM based on the FFT and IFFT is hampered by the difference in the direct and inverse algorithms.

Let us show that the use of the HT in OFDM systems not only is adequate for the use of the FT but also has certain advantages. A multifrequency signal can be represented as

$$s(t) = \sum_{n=0}^{N-1} c(f_n)\cos(2\pi f_n t). \quad (34)$$

Let us introduce the phase shift π/4 and the scaling factor $\sqrt{2}$ into Eq. (34). This yields

$$s(t) = \sqrt{2}\sum_{n=0}^{N-1} c(f_n)\cos(\pi/4 - 2\pi f_n t). \quad (35)$$

Obviously, the factor $\sqrt{2}$ changes only the amplitudes of harmonics and the phase shift π/4 does not distort frequency division multiplexing. If the trigonometric identity $\sqrt{2}\cos(\pi/4 - z) = \cos(z) + \sin(z) = \text{cas}(z)$ is taken into account and expressions (34) and (35) are used, the expression for multifrequency signal $s(t)$ can be represented in the form

$$\begin{aligned} s(t) &= \sum_{n=0}^{N-1} c(f_n)\text{cas}(2\pi f_n t) \\ &= \sum_{n=0}^{N-1} c(f_n)[\cos(2\pi f_n t) + \sin(2\pi f_n t)]. \end{aligned} \quad (36)$$

It follows from (1) that expression (36) is the HT. The orthogonality of channels can be proved easily by using the integral

$$\int_{iT_S}^{(i+1)T_S} c(f_i)c(f_j)\text{cas}(2\pi f_i t)\text{cas}(2\pi f_i t)dt = 0, \quad i \neq j.$$

Expression (36) describes group signal $s(t)$ with frequency division into N channels and cas-type orthogonal carrier frequencies. In order to recover the signal $c(f_n)$ on the temporal interval $[iT_s, (i+1)T_s]$, it is necessary to use expression (2). In that case, we obtain

$$\hat{c}(f_n) = \int_{iT_s}^{(i+1)T_s} r(t)\text{cas}(2\pi f_n t)dt. \quad (37)$$

Thus, expressions (36) and (37) imply that a signal is transmitted using the real kernel of the HT rather than the complex kernel of the FT as a carrier. The real kernel of the HT provides for computations that involve arithmetical operations with only real numbers and allows performance of direct and inverse transformations according to the same algorithm. When the HT is applied in the case of a real input data array, it is not necessary to form a special array by way of (33) because the Hartley spectrum remains real.

It follows from (36) that Hartley-basis transmission, combined with quadrature processing, necessitates one real summation operation and two real operations of multiplication by the corresponding function for each data-transmission channel. It is evident that the use of the FT involves two real summations and four real multiplications. Thus, for one channel, two multiplications and one summation are saved. This is equivalent to a computational gain of $2N$ real multiplications and N real summations for the algorithm on the whole.

Let us apply the DHT to construct transceivers. This technique allows application of the FHT. To this end, we split the temporal interval $[iT_s, (i+1)T_s]$ into N equal subintervals and represent (3) and (4) as

$$\begin{aligned} s(k) &= N^{-1}\sum_{n=0}^{N-1} c(n)\text{cas}(2\pi nk/N) \\ &= \text{FHT}\{c(0), c(1), \ldots, c(N-1)\}, \end{aligned} \quad (38)$$

$$\begin{aligned} \hat{c}(n) &= \sum_{k=0}^{N-1} r(k)\text{cas}(2\pi nk/N) \\ &= \text{FHT}\{r(0), r(1), \ldots, r(N-1)\}. \end{aligned} \quad (39)$$

The FHT is described in [2, 4, 7]. As in the case of the FFT, the computational complexity of the FHT can be assessed by quantity $N\log_2 N$. However, this quantity corresponds to the number of real rather than complex arithmetical operations. Thus, the FHT ensures a gain of approximately two as compared to the FFT, all other factors being the same. In addition, it follows from expressions (38) and (39) that both the transmitter and receiver use the same FHT algorithm; hence, the transceiver can be unified.

Thus, in this paper, modified algorithms have been developed for calculation of digital convolutions, cross-correlation functions, and autocorrelation functions in the spectral domain. These algorithms allow reduction of the computational work and required RAM by a third. This gain is obtained owing to the fact that the proposed algorithms necessitate only calculation of the Hartley spectra of the functions to be convolved and linear combinations of these spectra instead of direct calculation of the components involved in the HT in the conventional algorithm. A multifrequency data-transmission technique based on HT has been proposed. With this technique, the computational complexity of the algorithms used to form channels with orthogonal functions in existing OFDM systems is nearly halved. This result is attainable because all operations in the transceiver are performed in the real Hartley basis rather than the Fourier exponential-function basis. The use of the HT in OFDM offers certain additional advantages. Thus, intricate permutations of the input array become unnecessary in the case of real input data and the direct and inverse transforms are performed according to the same algorithm. Since the Hartley basis is orthogonal, this data-transmission technique is equivalent to the existing OFDM.

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